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This report provides interim documentation of a research project addressed to three aspects of nonlinear structural analysis: (i) Basic formulations of multibody system's equations of motion; (ii) A Lyapunov approach to designing globally stable feedback control laws; and (iii) symbol manipulation methods for deriving nonlinear control laws.


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Interim Report
for the period
July 1, 1989
through
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August 1990


Prepared by:
J. L. Junkins
A. J. Kurdila
Z. H. Rahman

Texas A&M Research Foundation
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# Contents

# SUMMARY


This document represents an interim report on research performed under Contract No. F49620-89-C-0084 from the Air Force Office of Scientific Research to Texas A&M University. The period covered by this report is from July 1, 1989 through July 31, 1990. The research results obtained are from the first year of a three year effort.

Significant progress is reported on analytical and computational methodology applicable to dynamics and control of flexible multibody structures structures. Especially significant are the following:

(i) We have developed new analytical and numerical results pertaining to imposing constraints in multi-body dynamical modeling and numerical simulation. We have developed an extension of existing penalty methods for constrained multibody dynamics, including some significant convergence proofs. These theoretical developments provide a rigorous foundation for existing penalty methods and lead directly to new methods. Of special significance are the analytical and numerical studies reported for linear substructuring.

(ii) We have developed a *power principle* which permits the efficient construction of stabilizing control laws for systems described by nonlinear systems of coupled ordinary and partial differential equations. It is not necessary to first spatially discretize the partial differential equations, and therefore this approach is immune to the many problems associated with truncation and spillover, vis-a-vis (for example) the validity of the stability boundaries. Preliminary, but very promising results have been obtained using this principle, including analytical, numerical and laboratory experiments.

(iii) We have initiated a study of symbol manipulation methods to derive polynomial-type nonlinear feedback control laws for dynamical systems with polynomial nonlinearities. A general MACSYMA symbolic computer code has been developed and studies are under way on several test problems. This work is in an early stage of development and the results obtained to date are mixed; we have found several attractive control laws, but for some systems surprising and as yet not understood failures to converge have occurred.

Since the current report documents the interim results from this study, we anticipate that the above results will have been significantly extended during the next year.


The Investigators for this effort were as follows: Professor J. L. Junkins served as Project Director and Principal Investigator. Professor A. J. Kurdila and Dr. Z. H. Rahman served as Co-Principal Investigators. Three Graduate Research Assistants (GRAs) participated in this project: N. Hecht, R. Menon, and S. Hsu. All three of these GRAs are pursuing their Ph. D. dissertation research and will likely be near completion by the time this effort is completed.

Organization of this report is a follows. We have presented the detailed technical results as attachments to this report. We have written the body of this report as a guided tour; following a brief introduction in Section 1, the sub-sections of Section 2 summarize several contributions with reference to the attachments. Section 3 provides concluding remarks and discusses some promising avenues for extending the research discussed in this report.

## 1.0 Introduction

Dynamics and control of structures research has recently entered an exciting era characterized by closer coordination of theoretical, computational, and experimental work. This trend is at least partially in response to the challenge to apply this emerging methodology to actual space structures within the next decade. A key to accelerating research progress, we believe, is that fundamental basic research needs to be blended with some constrained degree of applied research within the same research effort. This allows some carefully designed "reality testing" to be carried out early, providing feedback to the researchers inventing the methodology, and to provide "what it all means" insight for other researchers and the structural dynamics and control community at large. It is in this spirit that we have undertaken the present research project, and therefore we iterate between basic theoretical/analytical studies and carrying out constrained, simplified applications to provide some benchmarks for evaluating the results. In this interim report, we include fundamental analytical results as well as some of the numerical/experimental evaluation studies we have done to date. The technical details are contained in the several attachments.

## 2.0 Technical Accomplishments

With reference to the attachments, we overview the research contributions. In Section 2.1, we discuss some significant new results on a generalized penalty method for constrained multibody dynamics. Using this approach, we show that substructure dynamical models can be married into a global model without starting over and/or the necessity of inverting large linear systems to eliminate lagrange multipliers or solve for a reduced set of coordinates. This approach has some significant advantages over existing approaches. New analytical convergence proofs and convergence bounds are presented in the attachments. In Section 2.2, we summarize results fr´ ɹn a method for designing robust, globally stable control laws for nonlinear distributed parameter systems using a power principle, We show how to use this method, in the attachments, to design globally stable control laws for near-minimum-time maneuvers of flexible spacecraft. In Section 2.3, we overview some recent work we have done on an approach to use computer symbol manipulation to derive optimal feedback control laws for nonlinear dynamical systems.

## 2.1 Lyapunov Stable Penalty Methods for Multi-Body Dynamics

While there are many reasons why the simulation of multiple flexible bodies proves such a formidable task (the existence of multiple time scales for rigid and flexible degrees of freedom, high dimensionality, a nonlinearly varying generalized mass matrix...) one fact remains after twenty years of research: no completely satisfactory method has been developed that efficiently and reliably accounts for the "differential-algebraic" nature of the governing equations. It is well known that the inclusion of constraints using (Langrange) multipliers can be the cause of severe numerical difficulties. Researchers in numerical analysis have investigated general integration schemes applicable to a wide class of differential algebraic equations [Petzold 1,2,3,4]. Other researchers in the field of multibody simulation have suggested specialized means of detecting and/or avoiding these inherent conditioning problems, but at a significant computational cost. These methods include the nullspace methods described in [12], the range space methods in [21], Gears method [7], a particular implementation of Kane's equations [24] and Baumgarte's methods [3]. Some of the most novel approaches to appear recently are the formulations derived by Park [20] and [Bayo 1,2,3]. These methods use a penalty approximation of the Lagrange multiplier terms.

### 2.1.1 Basic Ideas

Penalty methods have been shown to be theoretically sound for certain classes of boundary value problems, such as the finite deformation of an incompressible elastic body [19]. However, "typical" hypotheses involved in the proofs associated with these such problems are that

> (i) the governing equations are obtained from the Gateaux differential of a *coercive functional (i.e., in some sense the functional satisfies a growth condition).*
>
> (ii) the Langrange multiplier terms satisfy a form of the Babuska-Brezzi condition.

It is not difficult to cite examples in multibody formulations of dynamics in which the first criterion is not satisfied. *But one should not conclude that penalty methods will not work for multibody dynamics formulations, only that the existing rigorous tools for proving existence and convergence of the method cannot be applied directly.*

The fact that penalty methods must be employed with care in application to on linear formulations of dynamics can be illuminated by a few simple linear examples. Consider first the dynamics of the free-free finite element model of a beam shown in figure (1). The usual means of obtaining the response of a clamped-clamped beam would be to use the constraint conditions and eliminate the redundant degrees of freedom, i.e., use a minimal set of generalized coordinates. Alternatively, one can retain redundant coordinates and use Lagrange multipliers to account for the constraint forces required to impose the constraints. *One means of obtaining an approximate solution to the problem using a penalty formulation is to simply adopt the estimate*

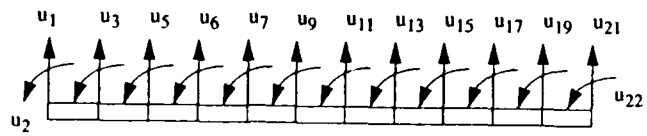



FIGURE (1) A 22 DOF FEM BEAM MODEL

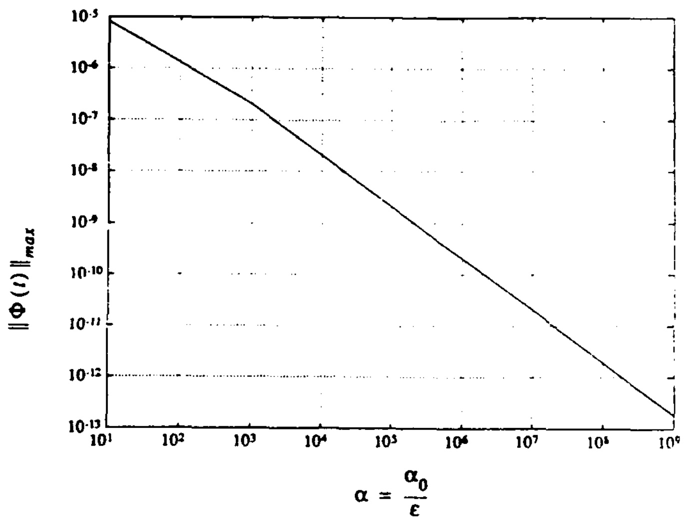



FIGURE (2) CONVERGENCE OF PENALTY METHOD

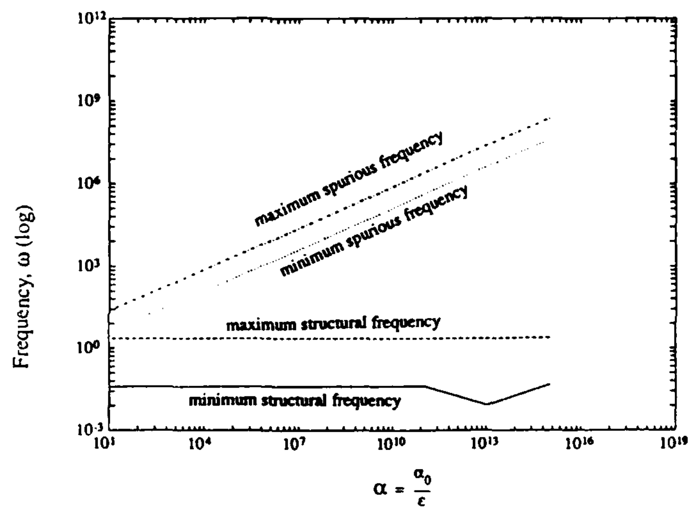



FIGURE (3) SPURIOUS FREQUENCIES : ZERO ORDER

$$\lambda = -\frac{1}{\varepsilon}\,\Phi$$

as is commonly employed in the simulation of constrained, coercive variational problems.

This procedure can lead to serious numerical difficulties in the transient problem as shown in figures (2) and (3). In this collection of problems, a 22-DOF free-free beam shown in figure (1) is clamped at the ends using the simple approximation above. In the first figure, the constraint violation at the ends of the beam is plotted as a function of the penalty parameter $\varepsilon$. Clearly, as the penalty parameter approaches zero, the constraint violation converges to zero as required. However, figure (3) shows that this approximation of the multipliers introduces spurious frequencies that become unbounded as the penalty term approaches zero. *The introduction of these extremely high, numerically introduced frequencies makes integration of the approximate governing equations by conditionally stable integration schemes impossible.* Just as importantly, an eigenvalue analysis of the method could reveal that the spurious frequencies are inter-mixed with the true structural frequencies. In some cases it is difficult to determine which frequencies are actual structural frequencies, and which are numerical artifacts. While [Park] and [Bayo1,2,3] have provided considerable empirical evidence that the penalty methods can be convergent and stable, there has been little analytical work to investigate these essential features of the formulations.

Throughout this research, the class of multibody systems under consideration consists of those that can be represented by Lagrange's equations

$$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_k}\right\} - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_k + \frac{\partial \phi_l}{\partial q_k}\,\lambda_l$$

where T and V are the kinetic and potential energies of the system, respectively, $q_k$ $k=1...N$ are the generalized coordinates, $Q_k$ are generalized forces and $\lambda_l$, $l = 1...D$ are the Lagrange multipliers. It is assumed that the constraint Jacobian matrix

$$\frac{\partial \phi_l}{\partial q_k}$$

has been derived from D holonomic constraints having the form

$$\phi_l\,(q_k) = 0$$

The dynamical system of Eqs. (1), (2), whose solution is the true motion of the system, is referred to as the "original, dynamical system."

The approach presented here approximates the dynamics of the original system; we introduced penalized potential and kinetic energies defined by

$$T_\varepsilon = T_\varepsilon\ (q_k, \dot{q}_k) = T + \frac{1}{2}\dot{\Phi}^T\ \beta\dot{\Phi}$$
$$V_\varepsilon = V_\varepsilon\ (q_k) = V + \frac{1}{2}\Phi^T\ \alpha\Phi$$

where $\alpha$ and $\beta$ are D X D symmetric, positive definite matrices and $\Phi$ is a D x 1 column vector of constraint functions. The subscript $\varepsilon$ is chosen such that the penalty matrices are parameterized by the variable $\varepsilon > 0$.

$$\alpha = \alpha\ (\varepsilon)$$
$$\beta = \beta\ (\varepsilon)$$

The penalty matrices are most simply selected to be diagonal with

$$\alpha_{ij} = \frac{\alpha_0}{\varepsilon}\ \varsigma_{ij} \qquad\qquad \beta_{ij} = \frac{\beta_0}{\varepsilon}\ \varsigma_{ij}$$

where

$$\alpha_0 \geq 0$$
$$\beta_0 \geq 0$$

One can also introduce a "Rayleigh constraint penalty dissipation function" via the definition

$$F_\varepsilon = \frac{1}{2}\ \dot{\Phi}^T\ \mu\ \dot{\Phi}$$

where $\mu$ is a D X D symmetric, positive definite matrix

$$\mu = \mu\ (\varepsilon)$$

with entries ordinarily, but not required to be, defined in the same manner as $\alpha$ and $\beta$

$$\mu_{ij} = \frac{\mu_0}{\varepsilon}\ \varsigma_{ij} \qquad\qquad \mu_0 \geq 0$$

The penalized potential and kinetic energies, and Rayleigh dissipation function are now used to generalize Arnold's single degree of freedom developments to obtain a system of equations suitable for the numerical simulation of multiple degree of freedom multibody systems.

$$\frac{d}{dt}\ \{\frac{\partial T_\varepsilon}{\partial \dot{q}_k}\} - \frac{\partial T_\varepsilon}{\partial q_k} + \frac{\partial V_\varepsilon}{\partial q_k} + \frac{\partial F_\varepsilon}{\partial \dot{q}_k} = Q_k$$

In the attachments [7.1-7.4] it is shown that these equations are kinematically equivalent to the *constraint violation feedback form* shown below.

$$\frac{d}{dt}\{\frac{\partial T_\varepsilon}{\partial \dot{q}_k}\} - \frac{\partial T_\varepsilon}{\partial q_k} + \frac{\partial V_\varepsilon}{\partial q_k} = Q_k - \frac{\partial \phi_i}{\partial q_k}\{\beta_{ij}\ \ddot{\phi}_j + \mu_{ij}\ \dot{\phi}_j + \alpha_{ij}\ \phi_j\}$$

Notice the qualitative affect of including the energy penalty terms and the Rayleigh constraint dissipation function is to invoke "feedback control" generalized forces proportional to the constraint violations and the first two derivatives thereof. It should be noted that the above formulation encompasses the starting point of the work by [Park], as well as the stiffness, damping and inertial methods of [Bayo].

We introduce a generalization of Arnold's convergence theorem:

*suppose*

(i) $$T(q_k,\dot{q}_k) = T_2(q_k,\dot{q}_k)$$

(ii) $$V = V(q_k)$$

(iii) $$\begin{cases} q_\varepsilon(0) = q(0) \\ \dot{q}_\varepsilon(0) = \dot{q}(0) \end{cases}$$

(iv) $$\Phi(q(0), \dot{q}(0)) = \Phi(q\ (0)) =$$

(v) $$\sigma_{min}(\alpha),\ \sigma_{min}(\beta) \leftrightarrow \infty \qquad as \quad \varepsilon \leftrightarrow 0$$

*then*

(i) $$E_\varepsilon = E(0) = T_\varepsilon + V_\varepsilon = T(0) + V\ (0)$$

(ii) $$||\dot{\Phi}_\varepsilon||^2 + ||\Phi_\varepsilon||^2 \leq \frac{2E(0)}{min(\sigma_{min}(\alpha),\ \sigma_{min}(\beta))} \leftrightarrow as\ \varepsilon \leftrightarrow 0$$

(iii) $$q_\varepsilon \leftrightarrow q \qquad as \qquad \Phi_\varepsilon \leftrightarrow 0$$

*where* $\sigma(\ )$ *denotes the singular values of* $(\ )$.

In addition to the most important convergence properties above, there also exist a wide class of systems for which the penalized governing equations are guaranteed to be stable in the sense of Lyapunov. Sufficient conditions for the stability of the penalized equations are summarized in the following theorem which has been extracted from our results in attachments [7.1-7.4].:

*Suppose*

$$(i) \qquad T(q_k, \dot{q}_k) = T_2(q_k, \dot{q}_k)$$

$$(ii) \qquad V = V(q_k) \quad \textit{is positive definite in the generalized coordinates}$$

$$(iii) \qquad \Phi(0) = \Phi(0) = 0$$

*Then a sufficient condition that the penalized governing equations are stable in the sense of Lyapunov is that*

$$-\Phi^T \mu \Phi = \sum_k Q_k \dot{q}_k \leq 0$$

*A sufficient condition that the penalized governing equations are asymptotically stable is that equality holds above only for*

$$\dot{q}_1 = \dot{q}_2 = \ldots = q_k = 0$$

$$q_1 = q_2 = \ldots = q_k = 0$$

The physical interpretation of this theorem is that the effects of the constraint penalty is to stabilize the open loop ($Q_k = 0$) system, since $\mu > 0$.

### 2.1.3 Imposing Constraints in Linear Substructuring via The Penalty Method

The above general results can be sharpened considerably for linear systems. In particular, an additional theoretical result exploits the analogy between the approximate penalty equations and symmetric, linear quadratic regulator feedback control. Suppose $\beta = 0$. In this case, the governing linear equations can be written

$$M\ddot{q} + C\dot{q} + Kq = -\left[\frac{\partial \Phi}{\partial q}\right]^T \{\mu\Phi + \alpha\Phi\}$$

$$M\ddot{q} + C\dot{q} + Kq = -\left[\frac{\partial \Phi}{\partial q}\right]^T \{\mu\left[\frac{\partial \Phi}{\partial q}\right]\dot{q} + \alpha\left[\frac{\partial \Phi}{\partial q}\right]q\}$$

By inspection the penalty terms can be identified as symmetric feedback control.

From Joshi [13], the symmetric, linear feedback for the linear system above is the *optimal feedback associated with the following performance index:*

$$J = \int_0^\infty \{X^T W X + 2X^T S U + U^T R U\}\, dt$$

or

$$J = \int [X^T U^T] \begin{bmatrix} W & S \\ S & R \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} dt$$

where X is the 2N vector

$$X = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

With the definition of weight matrices shown in the attachments (7.1-7.4), the performance index becomes

$$J = \int_0^\infty (2\Phi^T \mu \Phi + 2\dot{q}^T C \dot{q})\, dt$$

In other words, when $\beta = 0$ and the system is undamped, the penalty method maximizes a positive measure of the rate at which the (desired-to-be-zero) constraint energy is dissipated. We can also use LQR methods to pursue an alternative path leading to the stability conclusions of section 2.4.2.

Frequency domain synthesis of substructure models is also possible with the penalty approach. The undamped eigenvalue problem associated with the linear system is shown in attachments [7.1-7.4] to be

$$\left[\left[K + \left[\frac{\partial \Phi}{\partial q}\right]^T \alpha \left[\frac{\partial \Phi}{\partial q}\right]\right] + \lambda \left[M + \left[\frac{\partial \Phi}{\partial q}\right]^T \beta \left[\frac{\partial \Phi}{\partial q}\right]\right]\right] \psi = 0$$

If one chooses

$$\alpha = diagonal\ (\frac{\alpha_0}{\varepsilon} \ldots \frac{\alpha_0}{\varepsilon})$$

$$\beta = diagonal\ (\frac{\beta_0}{\varepsilon} \ldots \frac{\beta_0}{\varepsilon})$$

$$P = \left[\frac{\partial \Phi}{\partial q}\right]^T \left[\frac{\partial \Phi}{\partial q}\right]$$

$$P^2 = P$$

where *the rows of the Jacobian matrix have been orthononormalized so that P is an orthogonal projection onto the space of admissable configurations,* the perturbed eigenproblem becomes

$$\{\varepsilon\ (K + \lambda_\varepsilon M) + (\alpha_o + \lambda_\varepsilon \beta_0)P\}\ \Psi_\varepsilon = 0$$

By introducing the abstract angle between a subspace and a vector as

$$\cos\ (P, \Psi_\varepsilon) = \frac{\| P \Psi_\varepsilon \|}{\| \Psi_\varepsilon \|}$$

an error bound on the convergence of the eigenproblem is derived (attachments [7.1-7.4]) to be

$$\varepsilon \geq \frac{| \alpha_o + \lambda_\varepsilon \beta_0 | \cos (P, \Psi_\varepsilon)}{\sigma_{min} (K + \lambda_\varepsilon M)}$$

The application of the approximate eigenvalue analysis described in the last section has been applied in attachments [7.1-7.4] to a two substructure system having 120 degrees of freedom shown in figure (4). Figures (5), (6) have been extracted from attachments [7.1-7.4] and summarize the accuracy of the spectral estimates for a range of penalty parameters varying over eight orders of magnitude. The first figure shows that the norm of the rms relative error in all 108 natural frequency approximations remains below $10^{-3}$ for all values of $\varepsilon$. Figure (6) shows that the approximate eigenvalues obtained using LSPM had very small relative errors from the true frequencies. These errors vary from roughly 3 or 4 digits to nearly 10 digits of accuracy for different values of the penalty parameter. Thus, the method is extremely robust with respect to the selection of the penalty parameter.

### 2.1.4 Imposing Constraints in Nonlinear Multibody Dynamics via the Penalty Method

Several simulations have shown that the LSPM are extremely effective for nonlinear multibody systems. Two significant features of the method make it especially attractive as a means of constraint stabilization.

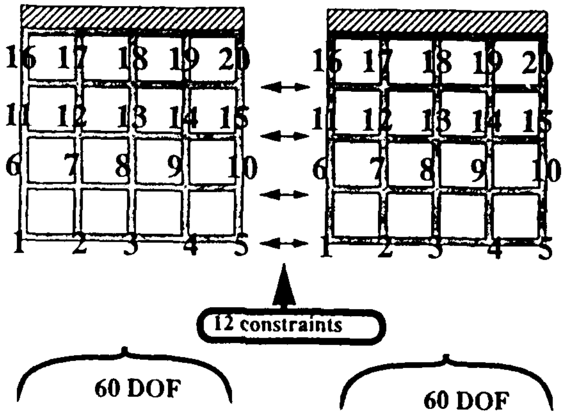



FIGURE (4) SUBSTRUCTURE EXAMPLE TEST GRID

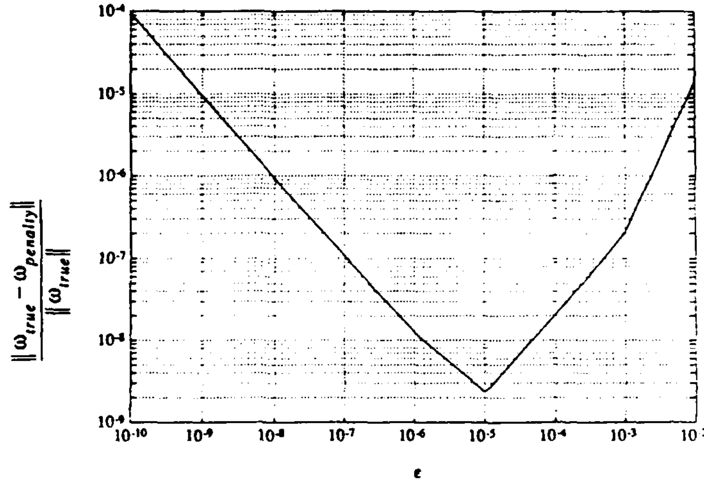



FIGURE (5) NORM OF THE SPECTRAL ERROR

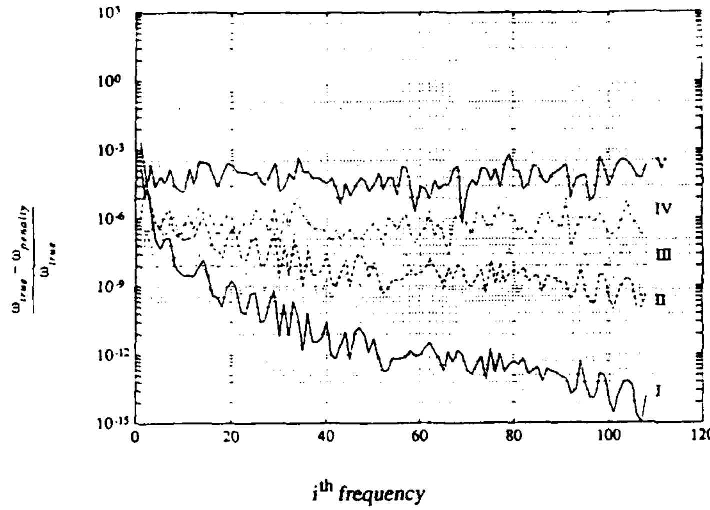



FIGURE (6) NORMALIZED ERROR IN EACH FREQUENCY

The first nonlinear example has been selected to show that the penalty method can be an attractive candidate for the simulation of systems having configuration dependent singularities. Such a typical system is the closed loop planar mechanism shown in figure (7). Figure (8) shows the time histories for $\Theta_1$ calculated by the range space method and penalty method, respectively. In this example the nonzero constraint violation damping term has been the selected to by $\gamma$ = 60000. As is evident from the diagram, the penalty method remains stable throughout the simulation, while the range space method eventually diverges at a singular configuration. The reason for the improved stability when the penalty method is employed is clear from the constraint violation plot (9). At each singular configuration, the energy transferred to the constraint degrees of freedom is rapidly dissipated, so that the norm of constraint violation returns to an acceptable level. It should be noted that this constraint violation stabilization is achieved at a cost: as the constraint energy is dissipated after each singular configuration the total system energy very slowly decreases, instead of remaining at the theoretically constant value. The author is currently investigating the use of energy-dissipation-rate-matching integration schemes, such as those in [5]. It is believed that this combination will prove to be a powerful tool for configuration singular problems.

As a general observation, it should be mentioned that it is not surprising that the penalty formulation described herein has "regularizing" characteristics; it resembles the method of Tikhonov regularization employed in singularity-robust pseudo-inverse problems [25].

The utility of the penalty method can be further illustrated in application to nonlinear multibody dynamics problems. Numerical simulations for a number of nonlinear, natural, conservative systems have verified the convergence and stability theorems derived earlier. However, the assumptions defined in the theorem above are somewhat limiting in that the convergence is guaranteed only for natural, conservative systems. Of particular interest to the authors are those multibody simulations in which one seeks to control the deployment of the system. In this section the qualitative behavior of the above numerical procedure for the simulation of dissipative systems is considered.

The example problem is shown in figure (10) is discussed further in attachments [7.1-7.4]. The essential features of the analysis are as follows:

(i) The convergence and stability theorems cited earlier guarantee that a Lyapunov function can be constructed for the penalized equations.

(ii) The global attractor of the constraint violation trajectories in the phase plane can be constructed using the Invariance Principle.

Simply put, this theorem asserts that the trajectory of a dynamical system, with a characterizing Lyapunov function defined on an open set G of phase space, must have an escape time, or approach the largest positive invariant subset M of

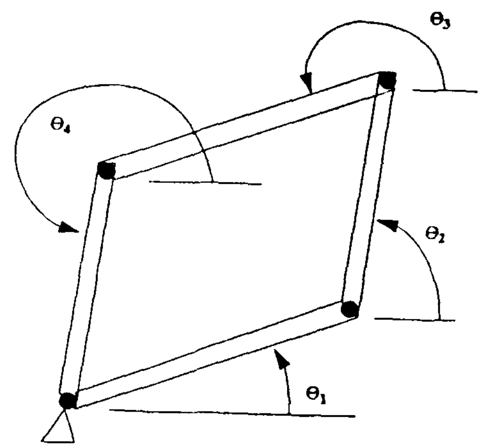



FIGURE (7) SINGULAR CONFIGURATION MECHANISM

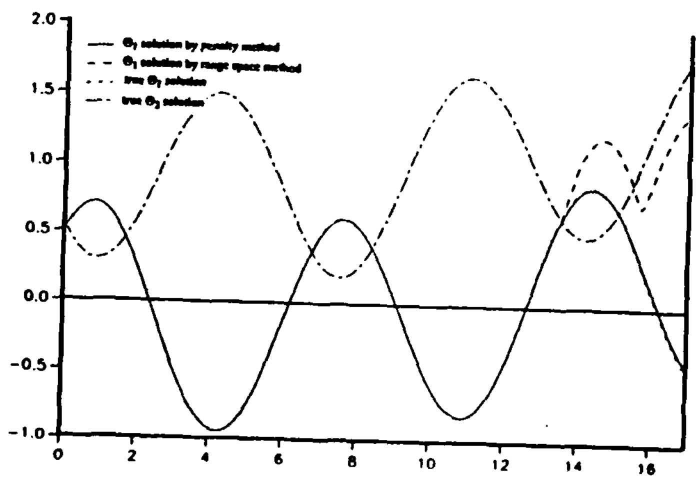



FIGURE (8) TIME HISTORY OF $\Theta_1$

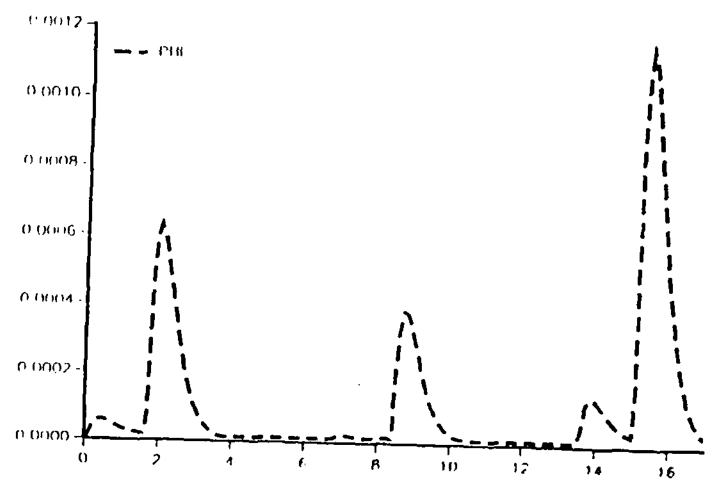



FIGURE (9) NORM OF CONSTRAINT VIOLATION

$$M \subset \{(q_k, \dot{q}_k) \in G \mid \dot{L} = 0\}$$

For the system considered, the global attractor is shown in the attachments [7.1-7.4] to be the family of ellipses given by

$$\frac{||\dot{\Psi}||^2}{\kappa/(m+\beta)} + \frac{||\Psi||^2}{\kappa/\alpha} = 1$$

where $\kappa$ is defined in the attachments [7.1-7.4]. Figure (11) verifies that the above equation does indeed characterize the limit cycle to which the constraint violation converges. In this case, $\alpha = \beta = 10000$.

Stronger conclusions can be obtained for this problem by noting that

$$\frac{\dot{\Psi}_1^2}{\kappa_1/(m+\beta)} + \frac{\Psi_1^2}{\kappa_1/\alpha} = 1$$

and

$$\frac{\dot{\Psi}_2^2}{\kappa_2/(m+\beta)} + \frac{\Psi_2^2}{\kappa_2/\alpha} = 1$$

together constitute a particular solution when (see attachments [7.1-7.4])

$$\kappa_1 + \kappa_2 = \kappa$$

The different limit cycles generated by specific ratios of $\alpha$ and $\beta$ are shown in the attachments. These limit cycles obviously exist for all choices of $\alpha$ and $\beta$, although the major and minor axes may be sufficiently small for a particular simulation if the penalty parameters can be made sufficiently large without encountering numerical difficulties. Evidently, if an initial (or numerically induced) energy spills into the constraint motion, then it theoretically remains in the invariant subset of the system. Geometrically, the path of the invariant motions for this example is a limit cycle corresponding to an elliptical translation (parallel displacement of the bar) without rotation. Note that only the rotation of the bar is controllable by the feedback torque; the "constraint degrees of freedom," which lie in the invariant subset, are uncontrollable via the dissipative feedback torque.

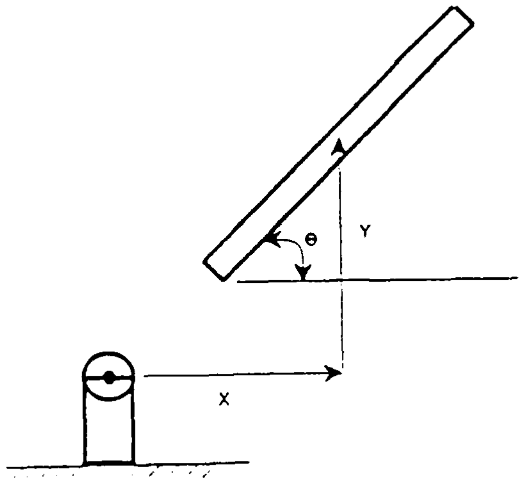



FIGURE (10) SIMPLE 3 DOF SYSTEM BEFORE CONSTRAINT

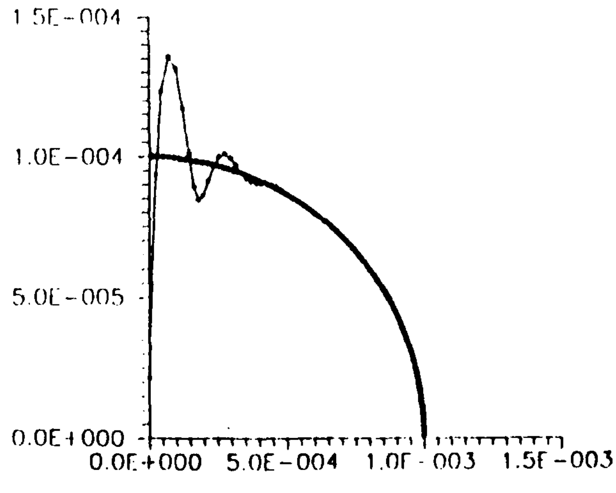



FIGURE (11) NORM OF Ψ LIMIT CYCLE, $\alpha = 100$, $\beta = 10000$

## 2.2 A Power Principle for Formulating Feedback Control Laws for Nonlinear Distributed Parameter Systems

### 2.2.1 Basic Ideas

Attachment 1 presents a novel approach we have developed for designing robust, globally stable control laws based upon a generalized energy-rate (power) principle. The approach is more akin to classical mechanics than control theory, however it elegantly extends the Lyapunov concepts used for stability analysis to establish a control law design method. The method will be seen to be rather democratic in that it applies to partial as well as ordinary differential equation models, and to nonlinear as well as linear models. It will also be seen to lead to highly robust control laws which guarantee stability for large families of modeling assumptions, rather than simple parametric or additive perturbations.

Suppose a structural system is acted upon by *m* actuators which can impart to-be-determined control forces (or moments) $\{u_1, u_2, \ldots, u_m\}$, the associated coordinates of the structural locations at which the actuator acts are the subset of system (linear or angular) coordinates $\{q_1, q_2, \ldots, q_m\}$. The first step in the development requires that the analyst introduce N+1 substructures for the purpose of describing the mechanical system's energy distribution by substructures. The system's error state (current position and velocity state minus the target state) is defined in terms of a *weighted error energy* function having the form:

$$U = \sum_{i=0}^{N} a_i E_i + a_{N+1} f(q_1, q_2, ..q_m, c_1, c_2, ....) \tag{2.2.1}$$

where $E_i = T_i + V_i$ is the total mechanical energy of the *ith* substructure, $T_i$ is the ith substructure's total kinetic energy, $V_i$ is the ith substructure's total potential energy, and $\{a_0, a_1, \ldots, a_{N+1}\}$ are the substructure energy weights; these weights will be seen below to parameterize the resulting feedback control laws. The function $f(q_1, q_2, \ldots, q_m, c_1, c_2, \ldots)$ is introduced for generality, because the total energy is not always positive-definite, and in particular, uncontrolled bodies often have rigid body freedoms which must be eliminated in order to make $U$ satisfy the fundamental necessary condition that it is a positive definite function having its global minimum at the target motion. Note that the $c_i$ denote free constants which may be chosen subject to the requirement that $f$ remain a non-negative function ot the *q's*. In the present form of Eq. (2.2.1), it is implicitly assumed that a zero energy rest state is the target motion, however, more generally, as developed in Attachment 1, the target state can be a prescribed reference motion. For introducing the ideas, we retain the simplest form of Eq. (2.2.1). Consider the total time derivative of the error energy function, this is obtained by differentiation of Eq. (2.2.1) to obtain

$$\dot{U} = \frac{dU}{dt} = \sum_{i=0}^{N} a_i \frac{dE_i}{dt}_i + a_{N+1} \sum_{j=1}^{m} \frac{\partial f}{\partial q_j} \dot{q}_j \tag{2.2.2}$$

The forces (and moments) acting on the ith substructure can be partitioned into four subsets, corresponding to:

(i) Forces having a potential $V_i$, with $V = \Sigma V_i$,

(ii) Forces which do no work on the ith sub structure (e. g., internal forces),

(iii) Boundary forces and moments acting on the ith substructure, and

(iv) The control force acting on the ith substructure.

Vectorially, the forces acting on the ith substructure can be written as

$$\boldsymbol{F}_i = -\vec{\nabla}V_i + \boldsymbol{F}_{non\text{-}working_i} + \boldsymbol{F}_{boundary\,forces_i} + B_i\boldsymbol{u}, \qquad \boldsymbol{u} = col\{u_1\ u_2\ \dots\ u_m\} \tag{2.2.3}$$

where $B_i$ is the control influence matrix, and $\boldsymbol{F}_{boundary\,forces_i}$ arises from interaction with adjacent substructures. From the work-energy principle (applicable with the same generality as Newton's Laws), we know that the change in kinetic energy of the ith substructure is given by the work/energy equation [26, 27, 28]

$$T_i - T_{o_i} = work_i = \sum_{\text{substructure}_i} \int_{t_o}^{t} \boldsymbol{F}_i \bullet \dot{R}_i\, dt \tag{2.2.4}$$

Substitution of Eq. (2.2.3) into (2.2.4) gives

$$E_i - E_{o_i} = \sum_{\text{substructure}_i} \left\{ \int_{t_o}^{t} \boldsymbol{F}_{boundary\,forces_i} \bullet \dot{R}_i\, dt + \int_{t_o}^{t} B_i \boldsymbol{u} \bullet \dot{R}_i\, dt \right\} \tag{2.2.5}$$

where $E_i = T_i + V_i$ is the total energy of the ith substructure. The developments of this section implicitly assume that the potential energy functions are non-negative, for the more general case modifications are required. Differentiation of Eq. (2.2.5) with respect to time gives the work-rate equation:

$$\frac{dE_i}{dt} = \sum_{\text{substructure}_i} \left\{ \boldsymbol{F}_{boundary\,forces_i} \bullet \dot{R}_i + B_i \boldsymbol{u} \bullet \dot{R}_i \right\} = power_i \tag{2.2.6}$$

Note that the above discussion can be easily generalized to accommodate boundary and control moments, we restrict the present discussion to forces for simplicity. A more general discussion is contained in the attachments. If we restrict the location of the control actuators to be at the boundaries of substructures, then for a natural system we can show that the total power of the ith substructure can be brought to the form

$$\frac{dE_i}{dt} = \sum_{j=1}^{m} \left( Q_{b_{ij}} + A_{ij} u_j \right) \dot{q}_j \tag{2.2.7}$$

where $Q_{b_{ij}}$ are the generalized forces associated with the boundary forces and moments, and $A_{ij}u_j$ is the jth generalized control force acting on the ith substructure. The $q_j$s are displacement coordinates of the actuator locations. Substitution of the substructure energy rates from Eq. (2.2.7) into Eq. (2.2.2) provides the following result for the Liapunov error energy rate of change:

$$\dot{U} = \sum_{j=1}^{m}\left[\sum_{i=0}^{N} a_i Q_{b_{ij}} + \sum_{i=0}^{N} a_i A_{ij} u_j + a_{N+1}\frac{\partial f}{\partial q_j}\right]\dot{q}_j \tag{2.2.8}$$

To guarantee that Eq. (2.2.8) is strictly non-positive, it is sufficient to require that each of the $m$ terms in the [ ] have a sign opposite to $\dot{q}_j$. The simplest choice is to simply set each of the [ ] terms to a negative constant ($-k_j$) times $\dot{q}_j$, this gives $\dot{U} = -\sum_{j=1}^{m} k_j \dot{q}_j^2$ and results in the following system of algebraic equations

$$\sum_{i=0}^{N} a_i Q_{b_{ij}} + \sum_{i=0}^{N} a_i A_{ij} u_j + a_{N+1}\frac{\partial f}{\partial q_j} = -k_j \dot{q}_j \tag{2.2.9}$$

These $m$ algebraic equations can be solved explicitly for the $m$ control functions and we can establish a stabilizing, constant gain, output feedback control law as

$$u_j = -\left(\frac{1}{\sum_{i=0}^{N} a_i A_{ij}}\right)\left[a_{N+1}\frac{\partial f}{\partial q_j} + k_j \dot{q}_j + \sum_{i=0}^{N} a_i Q_{b_{ij}}\right], j = 1, 2, \ldots, m \tag{2.2.10}$$

Note that the above control law feeds back measurements of position (through the dependence of $f$ on the $q$'s ), velocity, and boundary forces $Q_{b_{ij}}$. Notice that the constant gains are parameterized as a function of the energy weights ($a_i$ ) and the constants ($k_j$ ) and ($c_j$ ). Of course, the freedom to feed back boundary forces is only an advantage if these can be measured or estimated (via strain gauges or load cells, for example). Observe that the boundary forces on adjacent substructures will have equal magnitude and opposite sign, resulting in the last sum in the bracket combining in pairs; for the special case of a "chain configuration", for example, the sum has the structure: $\sum_{i=0}^{N} a_i Q_{b_{ij}} = (a_0 - a_1)Q_{b_{1j}} + (a_1 - a_2)Q_{b_{2j}} + \ldots$, thus, it is possible to place constraints on the substructure's energy weights (in this case, $a_{i-1} = a_i$) to either allow or eliminate selected boundary force feedback terms; note that the details of this summation is a function of the system topology and must be carried out specifically for each application. If it can be shown for the particular system that the generalized actuator location velocities $\{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_m\}$ vanish identically only at the target state, then it is evident that $\dot{U} = -\sum_{j=1}^{m} k_j \dot{q}_j^2$ is globally negative and we therefore have global asymptotic stability of the closed loop system.

The above discussion may appear a little abstract, but as is evident in the presentation of recent results below, it can be applied in a fairly straightforward way to achieve some very attractive control laws.

### 2.2.2 Discussion of Recent Results

With reference to figures 2.1 and 2.2, we consider specializing the above results for a particular multi-body maneuver problem. The nine body configuration and modeling assumptions are

## 2.2 Solution of Problem I

We seek an optimal feedback control law to compute $u(x)$ for

$$\dot{x} = -x + \alpha x^2 + u\,, \quad \alpha > 0 \tag{2.2.29}$$

which minimizes the performance measure $$J = \frac{1}{2}\int_0^{t_f} (x^2 + u^2)\, dt \tag{2.2.30}$$

The Hamiltonian for this system is

$$H = \frac{1}{2}(x^2 + u^2) + \lambda(-x + \alpha x^2 + u) \tag{2.2.31}$$

The Pontryagin necessary conditions are: $$\left\{\begin{aligned} &\frac{\partial H}{\partial u} = 0 \Rightarrow \quad u = -\lambda \\ &\dot{\lambda} = -\frac{\partial H}{\partial x} = -x + \lambda - 2\alpha x\lambda \\ &\dot{x} = \frac{\partial H}{\partial \lambda} = -x - \lambda + \alpha x^2 \end{aligned}\right\} \tag{2.2.32}$$

We seek a feedback form of the control law, specifically, we seek the polynomial gains $K_i$ in

$$-u = \lambda = \sum_{i=1}^{N} K_i x^i \tag{2.2.33}$$

Substituting Eq. (2.2.33) and its time derivative into Eqs. (2.2.32), we find the homogeneous condition

$$[\dot{K}_1 - 2K_1 - K_1^2 + 1]x + [\dot{K}_2 - 3(1+K_1)K_2 + 3\alpha K_1]x^2 +$$
$$[\dot{K}_3 - 4(1+K_1)K_3 + 4\alpha K_1 - 2K_2^2]x^3 + \dots + [\dot{K}_N - (N+1)(1+K_1)K_N - F_N(\alpha, K_1, \dots, K_{N-1})]x^N = 0$$

Since the above equation must hold at every point in the state space (i. e. , for all $x$), we conclude that all [ ]'ed coefficients must vanish independently, this provides the following equations:

$$\begin{aligned}
&\dot{K}_1 - 2K_1 - K_1^2 + 1 = 0 && \Rightarrow \quad K_1 \\
&\dot{K}_2 - 3(1+K_1)K_2 = -3\alpha K_1 && \Rightarrow \quad K_2 \\
&\dot{K}_3 - 4(1+K_1)K_3 = -4\alpha K_1 + 2K_2^2 && \Rightarrow \quad K_3 \\
&\qquad\vdots \\
&\dot{K}_N - (N+1)(1+K_1)K_N = F_N(\alpha, K_1, K_2, \dots, K_{N-1}) && \Rightarrow \quad K_N
\end{aligned} \tag{2.2.34}$$

Notice the following structure and properties of Eqs. (2.2.34) satisfied by the optimal gains:

- These equations can be solved *sequentially* for the $K_i$, all but the first are linear equations.
- The equations can solved to arbitrary order, since we have developed explicit recursions for $F_N$.
- The first equation for $K_1$, is a scalar Riccati equation (no surprise here!).
- If we impose $\alpha = 0$ and the boundary conditions $K_i(t_f) = 0$ [consistent with $\lambda(t_f) = 0$ as a transversality condition for $x(t_f)$ free], we see that all nonlinear gains vanish identically and therefore the above is indeed a direct generalization of the classical linear regulator with a quadratic performance index, the optimal control of Eq. (2.2.33) reduces to $u = -K_1 x$.
- If $t_f \to \infty$, analogous to the classical steady state regulator, we can show that all $K_i$ approach constants and Eqs. (2.2.34) therefore reduce to a sequence of algebraic equations for the constant gains (we can show that the positive real root is the proper selection for $K_1$), and the solution for the higher gains involves only simple algebraic operations. For this case, with $\alpha = 1$, the first few gains are numerically:
$$K_1 = 0.41321,\quad K_2 = 0.29289,\quad K_3 = 0.17678,\quad K_4 = 0.088388,\quad K_5 = 0.033145.$$

Shown below in Figure 2.3 is the performance index versus the order $N$ of the feedback control and the trajectories of $x(t)$ and $u(t)$ for typical initial conditions [$x(0)=1.3$]. As is qualitatively evident, the nonlinear terms are constructive and convergence is rapid. We have shown that the nonlinear controls for N>3 are globally stable, whereas large but finite domains of stability are associated with the linear and quadratic feedback control laws. This solution has been verified by the symbolic manipulator code. As we indicate below, the structure of this simplest scalar example fully generalizes, so that the above observations apply in a much more general context. However, the stability and convergence issues are problem-dependent, as should be expected. We have established a means to generate the matrix equivalent of the sequence of scalar equations ( Eqs. (2.2.34 ) and their solutions for the optimal nonlinear feedback gains

**Solution of Problem II**

We seek an optimal feedback control law to compute $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ for the system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 + x_2^2 + u_1 \\ \dot{x}_2 &= -x_2 + x_1 x_2 + x_1^2 + u_2 \end{aligned} \tag{2.2.35}$$

which minimizes

$$J = \frac{1}{2}\int_0^\infty \sum_{i=1}^{2} (x_i^2 + u_i^2)\, dt = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)\, dt \tag{2.2.36}$$

Eqs. (2.2.35) can be written as $\quad \dot{x} = A_1 x_1 + A_2 x_2 + Bu, \qquad Q = R = I \qquad (2.2.37)$

**Figure 2.1 A Nine Body Configuration**

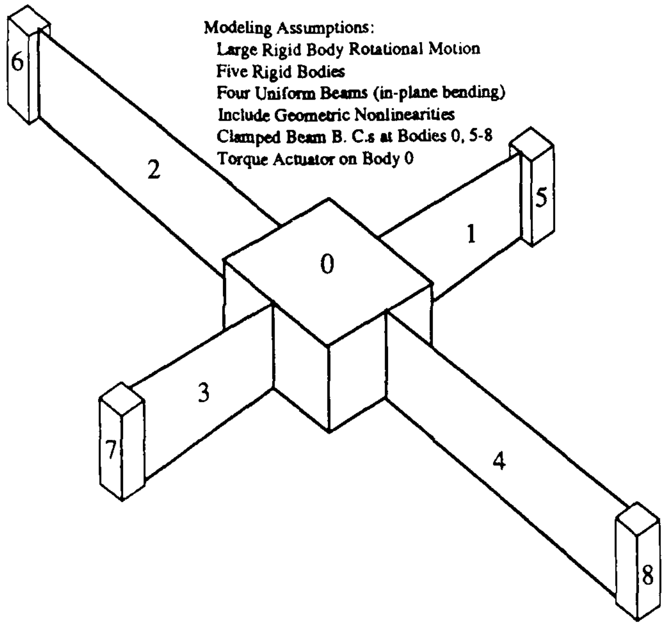
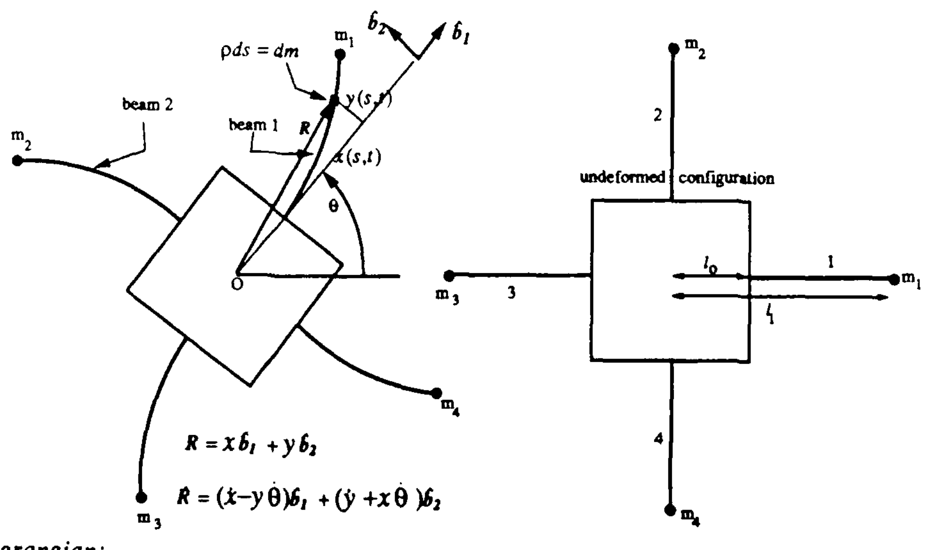



**Figure 2.2 Geometric/Kinematic Notations for the Equations of Motion**




System Lagrangian:

$$L = T - V = T_{hub} + T_{beam} + T_{tip} - V_{beam} = \frac{1}{2} I_{hub}\dot{\theta}^2 + \sum_{i=1}^{4}\left[\frac{1}{2}\int_{l_0}^{l_i} \dot{R}\cdot\dot{R}\rho_i \, ds_i + \frac{1}{2} m_i \dot{R}_{tip_i}\cdot\dot{R}_{tip_i} - \frac{1}{2}\int_{l_0}^{l_i} EI_i\left(\frac{\partial^2 y_i}{\partial s_i^2}\right)^2 ds_i\right] + \ldots$$

*Apply the Extended Hamilton's Principle*: $\int_{t_1}^{t_2}(\delta L + \delta W)dt + BCs \Rightarrow$ *PDE equations of motion* $\Rightarrow$

shown in figure 2.1, and the notation for the motion variables are given in figure 2.2. We seek to carry out large angle rotational maneuvers and vibration arrest with a single control torque $u(t)$ input which acts on body 0 about the vertical axis. The resulting hybrid system of ordinary and partial differential equations of motion are

$$I_{hub}\frac{d^2\theta}{dt^2} = u + \sum_{i=1}^{4}\mu_i\ , \quad \mu_i \equiv (M_o - S_o l_o)_i$$

$$\mu_i = -\int_{l_o}^{l_i}\rho_i x_i\left(\frac{\partial^2 y_i}{\partial t^2} + x_i\frac{d^2\theta}{dt^2}\right)dx_i + m_i\left(l_i\frac{d^2\theta}{dt^2} + \frac{\partial^2 y_i}{\partial t^2}\Big|_{l_i}\right) + HOT \tag{2.2.11}$$

$$\rho_i\left(\frac{\partial^2 y_i}{\partial t^2} + x_i\frac{d^2\theta}{dt^2}\right) + \mathrm{EI}\frac{\partial^4 y_i}{\partial x_i^4} = 0 + HOT\ , \quad i = 1, 2, 3, 4$$

*HOT* indicates other *known* linear & nonlinear effects (such as rotational inertia effects, rotational stiffening, foreshortening effects, shear deformation, etc.). The boundary conditions are

$$\begin{aligned} &at\ x_i = l_o:\ y_i(t, l_o) = 0, && \frac{\partial y_i}{\partial x}\Big|_{l_o} = 0 \\ &at\ x_i = l_i:\ \frac{\partial^2 y_i}{\partial x_i^2}\Big|_{l_i} = 0 + HOT, && \frac{\partial^3 y_i}{\partial x_i^3}\Big|_{l_i} = \frac{m_i}{\mathrm{EI}_i}\left(l\frac{d^2\theta}{dt^2} + \frac{\partial^2 y_i}{\partial t^2}\Big|_{l_i}\right) \end{aligned} \tag{2.2.12}$$

The total energy of the system (constant in the absence of control or disturbances) is:

$$2E = I_{hub}\left(\frac{d\theta}{dt}\right)^2 + \sum_{i=1}^{4}\left[\int_{l_o}^{l_i}\rho_i\left(\frac{\partial y_i}{\partial t} + x_i\frac{d\theta}{dt}\right)^2 dx_i + \int_{l_o}^{l_i}EI_i\left(\frac{\partial^2 y_i}{\partial x_i^2}\right)^2 dx_i + m_i\left(l_i\frac{d\theta}{dt} + \frac{\partial y_i}{\partial t}\Big|_{l_i}\right)^2\right] \tag{2.2.13}$$

In view of the above energy integral, we investigate the Liapunov function

$$2U = a_0 I_{hub}\left(\frac{d\theta}{dt}\right)^2 + \sum_{i=1}^{4}a_i\left[\int_{l_o}^{l_i}\rho_i\left(\frac{\partial y_i}{\partial t} + x_i\frac{d\theta}{dt}\right)^2 dx_i + \int_{l_o}^{l_i}EI_i\left(\frac{\partial^2 y_i}{\partial x_i^2}\right)^2 dx_i + m_i\left(l_i\frac{d\theta}{dt} + \frac{\partial y_i}{\partial t}\Big|_{l_i}\right)^2\right] + a_5(\theta - \theta_f)^2 \tag{2.2.14}$$

The positive weighting coefficients $a_i > 0$ allow relative emphasis upon five substructures' contributions to the total *error energy* of the system. Note that the open loop system energy integral of Eq. (2.2.13) does not depend upon the rigid body displacement, *the final term is introduced so that Eq. (2.2.14) has its global minimum at the target final state*:

$$\{\theta, \dot{\theta}\}_{desired} = (\theta_f, 0)\ , \qquad \left\{y_i(x_i, t), \frac{\partial y_i(x_i, t)}{\partial t}\right\}_{desired} = (0, 0), \quad i = 1, 2, 3, 4$$

More generally, error energy can be measured from a time varying *target trajectory* [j1].

Differentiation of Eq. (2.2.4), substitution of the equations of motion (Eqs. (2.2.1), (2.2.2)), and substantial calculus leads to

$$\dot{U} = \frac{dU}{dt} = \dot{\theta}\left[\, a_0 u + a_5(\theta-\theta_f) + \sum_{i=1}^{4}(a_i - a_0)\mu_i \,\right] \tag{2.2.15}$$

Alternatively, and much more efficiently, we could have written Eq. (2.2.15) down immediately using free body diagrams and Eq. ( 2.2.8).

Since we require that $\dot{U} \le 0$, we set the [ ] term to $-a_6\dot{\theta}$ and this leads to $\dot{U} = -a_6\dot{\theta}^2$ and the control law:

$$u = -\frac{1}{a_0}\left[\, a_5(\theta-\theta_f) + a_6\dot{\theta} + \sum_{i=1}^{4}(a_i - a_0)\mu_i \,\right] \tag{2.2.16}$$

which, again, we could have written directly from Eq. (2.2.10). Thus we see that the following *linear, spatially discrete* output feedback law satisfies the *sufficient condition* ($\dot{U} \le 0$) to globally stabilize this distributed parameter system:

$$u = -\left[\, g_1(\theta-\theta_f) + g_2\dot{\theta} + \sum_{i=3}^{6} g_i\mu_i \,\right],\quad g_1 \equiv \frac{a_5}{a_0} \ge 0,\quad g_2 \equiv \frac{a_6}{a_0} \ge 0,\quad g_i \equiv \frac{a_{i-2}-a_0}{a_0} \ge -1 \tag{2.2.17}$$

The *pervasive dissipation condition that* $\dot{U} = -a_6\dot{\theta}^2$ *is strictly negative*, for asymptotic stability, is satisfied *only if the system is fully controllable*. In the linear case, we find that the *anti-symmetric in opposition* modes, (for a perfectly symmetric structure, 4 identical appendages) have zero hub motion & are uncontrollable by a hub torque actuator -- however, these modes are also theoretically (assuming perfectly identical appendages and clamped boundary conditions) un-disturbed for rest-to-rest maneuvers using a hub actuator [26]. It is of significance that we have proven that the same results [Eqs. (2.2.15) - (2.2.17)] are obtained when we generalize the physical modeling assumptions above to include any/all of the following effects: (1) shear deformation and rotary inertia, (2) rotational stiffening and foreshortening effects, (3) any/all positive semi-definite functionals modeling the mechanical potential energy storage associated with beam deformation.

The invariance of the *form* for the stabilizing control law and stable gain region, with respect to the most common variations in modeling assumptions, represents an important generalization of the well known robustness obtained using only the positive local velocity feedback term in Eq. (2.2.17). The physical source of this robustness is the truth that the energy rate is always given by Eq. (2.2.6), irregardless of whether or not we have correctly modeled the actual system's physics. Therefore we have essentially restricted the discussion to control laws which cause the error energy to decrease. The set of system physical models which will be stabilized at the target state are such that the physically correct energy substructure functionals $E_i$, when substituted into Eq. (2.2.1), yield a positive definite functional with its global minimum at the target state. Thus by rigorously stabilizing a large family of system models by the same control law, we are then left to hope that the set of stabilized models contains our actual system's physics. It is fortuitous that almost all beam constituitive modeling assumptions indeed result in a positive beam potential energy functional which vanishes for zero deformation, and obviously, zero deformation is the most common target state! Of course the accuracy of the predicted closed loop response (while stability may be assured) is indeed a function of the degree to which the system is accurately modeled, and clearly the optimization of the control gains is model-dependent.

For the special case for which the four appendages and tip masses are assumed identical, we have studied the above control law and the generalizations thereof analytically, numerically, and have conducted successful laboratory experiments. [see reference 26 and Attachment 3]

### 2.2.3 Status and Outlook

As is evident from the above discussion and Attachment 1, we have made significant progress on several fronts: (1) development of a novel approach for designing stable control control laws for nonlinear distributed parameter systems, (2) analytical and numerical studies of the validity of the approach for special cases, and (3) laboratory experimental validation of the approach for a simple multi-body maneuver configuration. The results to date have been very encouraging. We have also addressed extensions of the basic methodology to permit tradeoffs between competing measures of optimality such as minimum maneuver time versus minimum vibration measures (see attachment 3). During the next year, we expect to devote most of our effort to three theoretical issues: (i) Addressing the issues raised by uncontrollable and/or poorly controllable dynamical sub-spaces of motion (e. g., for a linear structure, uncontrollable natural vibration modes) for nonlinear systems as a function of actuator configurations. (ii) Extending/modifying the power principle to consider quasi coordinate descriptions of the system kinematics. (iii) Extending/modifying the power principle to address *non-natural* systems for which the kinetic energy is not a symmetric quadratic form in the generalized velocities, and the potential energy is not a positive definite function of the generalized coordinates. All three of these theoretical issues will be studied in the light of simple example structures, motivated by potential practical applications, with appropriate characteristics to illuminate the salient features of the developments.

## 2.3 Symbolic Derivation of Nonlinear Feedback Control Laws

### 2.3.1 Basic Ideas

In a somewhat unorthodox, but we trust, effective format, we motivate the symbolic nonlinear control design approach by first presenting several example nonlinear control problems then their solutions which we have recently been able to carry to completion. We first state the three problems, then outline the main features of their solutions. The first two problems are overly "academic" examples selected because they offer a transparent way to introduce these ideas. The third problem has obvious practical significance, this example illustrates the use of this approach to design an optimal feedback control law for the nonlinear spacecraft attitude maneuver problem. These ideas grew from our historical research documented in references [29, 30].

**Three Problem Statements**

To illustrate the concepts, we introduce the differential equations and performance indices for three nonlinear control problems. The first two problems are provided for a simple introduction, but they have a similar structure to the third problem (large angle, nonlinear spacecraft attitude maneuvers), and the same methodology readily solves all three problems.

**Problem I**

Find an optimal feedback control law to compute $u(x)$ for the system described by

$$\dot{x} = -x + \alpha x^2 + u\,, \quad \alpha > 0 \tag{2.2.18}$$

which minimizes the performance measure

$$J = \frac{1}{2}\int_0^{t_f} (x^2 + u^2)\, dt \tag{2.2.19}$$

**Problem II**

Find an optimal feedback control law to compute $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ for the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 + x_2^2 + u_1 \\ \dot{x}_2 &= -x_2 + x_1 x_2 + x_1^2 + u_2 \end{aligned} \tag{2.2.20}$$

which minimizes the performance measure $$J = \frac{1}{2}\int_0^{t_f} \sum_{i=1}^{2} (x_i^2 + u_i^2)\, dt \tag{2.2.21}$$

Note that Eqs. (2.2.20) can be written in an alternate matrix format as

$$\dot{\mathbf{x}} = A_1 \mathbf{x}_1 + A_2 \mathbf{x}_2 + B\mathbf{u}\,, \tag{2.2.22}$$

with $\mathbf{x}_1^T = \mathbf{x}^T = [x_1 \;\; x_2]$, $\mathbf{x}_2^T = [x_1^2 \;\; x_1 x_2 \;\; x_2^2]$, $\mathbf{u}^T = [u_1 \;\; u_2]$, $A_1 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

As we show below, the above notation generalizes to encompass high dimensional nonlinear systems. The linear appearance of the above notation should not obscure the fact that the system of Eq. (2.2.22) is nonlinear, note that the vector $\boldsymbol{x}_2$ contains all of the quadratic nonlinear terms.

**Problem III**

Find an optimal feedback law for the control torques $u_i(q_1, q_2, q_3, \omega_1, \omega_2, \omega_3)$, for $i = 1,2,3$ to control the six-state, three-input system described by [ 30] the following system of equations

$$\begin{aligned}
\dot{q}_1 &= \tfrac{1}{2}[(1+q_1^2)\omega_1 + (q_1 q_2 - q_3)\omega_2 + (q_1 q_3 + q_2)\omega_3] \\
\dot{q}_2 &= \tfrac{1}{2}[(q_1 q_2 + q_3)\omega_1 + (1+q_2^2)\omega_2 + (q_2 q_3 - q_1)\omega_3] \\
\dot{q}_3 &= \tfrac{1}{2}[(q_1 q_3 - q_2)\omega_1 + (q_2 q_3 + q_1)\omega_2 + (1+q_3^2)\omega_3] \\
\dot{\omega}_1 &= \left(\tfrac{I_2 - I_3}{I_1}\right)\omega_2\omega_3 + \left(\tfrac{1}{I_1}\right)u_1 \\
\dot{\omega}_2 &= \left(\tfrac{I_3 - I_1}{I_2}\right)\omega_3\omega_1 + \left(\tfrac{1}{I_2}\right)u_2 \\
\dot{\omega}_3 &= \left(\tfrac{I_1 - I_2}{I_3}\right)\omega_1\omega_2 + \left(\tfrac{1}{I_3}\right)u_3
\end{aligned} \tag{2.2.23}$$

which minimizes the performance index $$J = \frac{1}{2}\int_0^\infty (\boldsymbol{x}^T Q \boldsymbol{x} + \boldsymbol{u}^T R \boldsymbol{u})\, dt \tag{2.2.24}$$

$Q$, and $R$ are symmetric positive definite weight matrices; the state and control vectors are

$$\boldsymbol{x}^T = [q_1\ q_2\ q_3\ \omega_1\ \omega_2\ \omega_3]\ ,\ \boldsymbol{u}^T = [u_1\ u_2\ u_3]$$

The $q$'s are the three Rodriguez attitude coordinates [28] and the $\omega$'s are the three orthogonal components of the angular velocity vector along principal axes. The $u$'s are the control torques about the principal axes. The $I$'s are the principal inertias. These equations govern the general, large angle, nonlinear attitude maneuvers of a rigid spacecraft. An advantage of the Rodriguez parameters, compared to any choice of three Euler angles, is that no transcendental functions appear. The above equations are exact, the degree of polynomial nonlinearity is three. Note that the target orientation of the body ($q_i = 0$) has been selected as the inertial frame.

Equations (6) can be viewed as a special case of the most general nonlinear differential equation

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{3} A_i \boldsymbol{x}_i + B\boldsymbol{u} \tag{2.2.25}$$

where $\boldsymbol{x}_i$ are column vectors containing the 6 linear state variables ($\boldsymbol{x} \equiv \boldsymbol{x}_1$), the 21 distinct quadratic combinations of the state variables ($\boldsymbol{x}_2$), and the 56 cubic combinations of the state variables ($\boldsymbol{x}_3$), for above case (as in all others studied to date!), the three $A_i$ matrices are very sparse, as given in detail below.

$$
\overset{6x1}{x}_1 = x = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}, \quad
\overset{21x1}{x}_2 = \begin{matrix} x_1^2 \\ x_1x_2 \\ x_1x_3 \\ x_1x_4 \\ x_1x_5 \\ x_1x_6 \\ x_2^2 \\ x_2x_3 \\ x_2x_4 \\ x_2x_5 \\ x_2x_6 \\ x_3^2 \\ x_3x_4 \\ x_3x_5 \\ x_3x_6 \\ x_4^2 \\ x_4x_5 \\ x_4x_6 \\ x_5^2 \\ x_5x_6 \\ x_6^2 \end{matrix}, \quad
\overset{56x1}{x}_3 = \begin{matrix} x_1^3 \\ x_1^2x_2 \\ x_1^2x_3 \\ x_1^2x_4 \\ x_1^2x_5 \\ x_1^2x_6 \\ x_1x_2^2 \\ x_1x_2x_3 \\ x_1x_2x_4 \\ x_1x_2x_5 \\ x_1x_2x_6 \\ x_1x_3^2 \\ x_1x_3x_4 \\ x_1x_3x_5 \\ x_1x_3x_6 \\ x_1x_4^2 \\ x_1x_4x_5 \\ x_1x_4x_6 \\ x_1x_5^2 \\ x_1x_5x_6 \\ x_1x_6^2 \\ x_2^3 \\ x_2^2x_3 \\ x_2^2x_4 \\ x_2^2x_5 \\ x_2^2x_6 \\ x_2x_3^2 \\ x_2x_3x_4 \\ x_2x_3x_5 \\ x_2x_3x_6 \\ x_2x_4^2 \\ x_2x_4x_5 \\ x_2x_4x_6 \\ x_2x_5^2 \\ x_2x_5x_6 \\ x_2x_6^2 \\ x_3^3 \\ x_3^2x_4 \\ x_3^2x_5 \\ x_3^2x_6 \\ x_3x_4^2 \\ x_3x_4x_5 \\ x_3x_4x_6 \\ x_3x_5^2 \\ x_3x_5x_6 \\ x_3x_6^2 \\ x_4^3 \\ x_4^2x_5 \\ x_4^2x_6 \\ x_4x_5^2 \\ x_4x_5x_6 \\ x_4x_6^2 \\ x_5^3 \\ x_5^2x_6 \\ x_5x_6^2 \\ x_6^3 \end{matrix},
$$

*The non-zero elements of the $A_i$ are:*

$$\overset{6x6}{A}_1(1,4) = \overset{6x6}{A}_1(2,5) = \overset{6x6}{A}_1(3,6) = \tfrac{1}{2}$$

$$\overset{6x21}{A}_2(1,11) = \overset{6x21}{A}_2(2,13) = \overset{6x21}{A}_2(3,5) = \tfrac{1}{2}$$

$$\overset{6x21}{A}_2(3,14) = \overset{6x21}{A}_2(2,6) = \overset{6x21}{A}_2(3,9) = -\tfrac{1}{2}$$

$$\overset{6x21}{A}_2(4,20) = \left(\frac{I_2-I_3}{I_1}\right)$$

$$\overset{6x21}{A}_2(5,18) = \left(\frac{I_3-I_1}{I_2}\right)$$

$$\overset{6x21}{A}_2(6,17) = \left(\frac{I_1-I_2}{I_3}\right) \tag{2.2.26}$$

$$\overset{6x56}{A}_3(1,4) = \overset{6x56}{A}_3(1,10) = \overset{6x56}{A}_3(1,15) = \tfrac{1}{2}$$

$$\overset{6x56}{A}_3(2,9) = \overset{6x56}{A}_3(2,25) = \overset{6x56}{A}_3(2,30) = \tfrac{1}{2}$$

$$\overset{6x56}{A}_3(3,13) = \overset{6x56}{A}_3(3,29) = \overset{6x56}{A}_3(3,37) = \tfrac{1}{2}$$

*The B matrix is:*

$$B = \begin{bmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & \frac{1}{I_3} \end{bmatrix}$$

Prior to presenting the solutions for the feedback controls for the above three problems, we state that we have developed a general approach which applies the optimal control necessary conditions (Pontryagin's Principle and Pontryagin's necessary conditions [28]), to minimize an index of the form of Eq. (2.2.24)) and generates *symbolically* the necessary conditions, and solves these for the particular set of symbolic differential and algebraic equations governing the optimal nonlinear feedback control gains for *any particular member of the class of nonlinear dynamical systems* described by differential equations of the form

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{M} A_i \boldsymbol{x}_i + B\boldsymbol{u} \qquad (2.2.27)$$

where $\boldsymbol{x}$ is an *nx1* state vector, $\boldsymbol{u}$ is an *mx1* control vector, $M$ is the highest degree of polynomial nonlinearity, $\boldsymbol{x}_i$ is a column vector containing all distinct polynomial combinations (of degree $i$) of the elements of the state vector $\boldsymbol{x}$. It is obvious that Eqs. (2..2.27) and the generalized problem statement embraces a very large class of systems, including all three of the above-stated problems as special cases, and therefore a large family of nonlinear mechanics problems arising in structural dynamics and control. The nonlinear feedback controls $\boldsymbol{u}$ are found as direct solutions of the general necessary conditions for optimal control of the general family of systems described by Eqs. (2.2.27). Our approach leads directly to symbolic equations satisfied by the optimal feedback gain matrices $G_i$ in the polynomial expansion

$$\boldsymbol{u} = \sum_{i=1}^{N} G_i \boldsymbol{x}_i, \text{ where } \quad G_i = -R^{-1} B^T K_i, \qquad \lambda = \sum_{i=1}^{N} K_i \boldsymbol{x}_i \qquad (2.2.28)$$

The degree $(N)$ of the nonlinear feedback control expansion is *not* restricted to be equal to the degree $(M)$ of the nonlinearity in the original differential equations (it is fortuitous, as is confirmed in the examples below, that $N$ typically required for practical convergence is in fact usually of low degree, but no universal conclusion can be made). Note that $\lambda$ is an *nx1* vector of Lagrange multipliers which arise in the optimal control necessary conditions [30].

It is significant to note that we can find, via symbol manipulation, a set of sequentially solvable, *general algebraic and/or differential equations* satisfied by the gains $K_i$ where the matrices $A_i$, $B$, $Q$, $R$ , or subsets thereof, can appear as *algebraic parameters* (i. e., we do not first have to first specify numerical values for the system parameters, or even numerical elements for the weight matrices); we can leave any subset (or all) of the system parameters as symbols and develop a sequence of equations which *are specialized for the size and sparsity patterns of the matrices of the particular system of interest*. These equations can then be solved numerically for the gains corresponding to the particular applications of interest. Another way of viewing our result is to observe that it has been known for several decades that the optimal linear control gains, for a linear system, are generated through solution of a matrix Riccati equation which depends explicitly upon matrices $A$, $B$, $Q$, $R$ ; we have conceived of a systematic way to develop the analogous, explicit, sequentially solvable equations governing the higher order nonlinear feedback gains, for the class of systems described by Eq. (12.2.27).

It is apparent that a general realization of this approach would be very attractive, not only because it makes determination of the equations governing the nonlinear gains relatively routine, but also because changes in subsets of the system parameters or weights in the performance measure can be quickly accounted for by simply changing the desired parameters in the (one-time-derived) gain equations and then simply re-solving these equations for the new gains. This approach would be quite impossible, for nonlinear systems of even moderate dimensionality, without use of modern computer algebraic manipulation. While a given dynamical system might submit after man-months of algebra and calculus, it would only rarely be implemented successfully due to the associated large investment of human effort, elapsed time, and the intimidating problems raised by debugging the results and integrating this process into an invariably iterative controller design cycle. It appears evident that a successful implementation of our approach will make possible routine application of perturbation methods to derive nonlinear control laws, at least for large families of problems (not merely *in principle*, but *in fact*!).

Our first implementation ( using *macsyma* ) of these ideas can be applied "fairly routinely" without requiring intolerable storage or execution time, if the product of $MN$ is less than about 20. Due to the several curses of dimensionality, we do not feel that this approach will prove practical in the near term if $MN$ is greater than about 40, but this is still encompasses a very large class of problems. In the applications to date, we have found the matrices involved and the resulting control gains are rather sparse, it is possible that the curse of dimensionality can be substantially reduced by introducing (as yet undeveloped) methods to anticipate and take advantage of the sparse structure of the particular problem. Immediate extensions to higher dimensions can be undertaken, but we feel that effects and problems associated with dimensionality increases should be studied carefully in the context of a systematic, escalator-styled research effort, taking time to consider specific applications. This will permit the evolving formulations and computer implementations to benefit fully from the insights which stem from using the methods on problems small enough to make the salient features transparent. Since this approach represents a new controller design methodology, the basic research will no doubt benefit from the analytical and artistic insights gained from several case study applications. Convergence proofs are not available for arbitrary systems belonging to the general class described by Eqs. (2.2.27 - 2.2.28), but convergence will be studied on a case-by-case basis, and we will seek to establish insights on how to approach resolving the convergence issue for high-dimensioned, high-order expansions.

In order to make the essential ideas easy to understand and to simultaneously display some of the progress we have made to date, we present the solution details for the first problem as we originally developed them *by hand*, these same results have been confirmed and extended using the symbolic manipulator. For the second and third problems, we provide only solution outlines and some of the 'end products' of this development, due to space limitations. All three of the above stated problems have been solved by hand (to low order) to verify the correctness of the symbol manipulation implementation, we have also successfully completed comparisons of the results with Carrington's dissertation [29]. We now discuss the solutions of the three problems.

## 2.2 Solution of Problem I

We seek an optimal feedback control law to compute $u(x)$ for

$$\dot{x} = -x + \alpha x^2 + u\,, \quad \alpha > 0 \tag{2.2.29}$$

which minimizes the performance measure $$J = \frac{1}{2}\int_0^{t_f} (x^2 + u^2)\, dt \tag{2.2.30}$$

The Hamiltonian for this system is

$$H = \frac{1}{2}(x^2 + u^2) + \lambda(-x + \alpha x^2 + u) \tag{2.2.31}$$

The Pontryagin necessary conditions are:
$$\left\{ \begin{array}{l} \dfrac{\partial H}{\partial u} = 0 \Longrightarrow \quad u = -\lambda \\ \dot{\lambda} = -\dfrac{\partial H}{\partial x} = -x + \lambda - 2\alpha x \lambda \\ \dot{x} = \dfrac{\partial H}{\partial \lambda} = -x - \lambda + \alpha x^2 \end{array} \right\} \tag{2.2.32}$$

We seek a feedback form of the control law, specifically, we seek the polynomial gains $K_i$ in

$$-u = \lambda = \sum_{i=1}^{N} K_i x^i \tag{2.2.33}$$

Substituting Eq. (2.2.33) and its time derivative into Eqs. (2.2.32), we find the homogeneous condition

$$[\dot{K}_1 - 2K_1 - K_1^2 + 1]x + [\dot{K}_2 - 3(1+K_1)K_2 + 3\alpha K_1]x^2 +$$
$$[\dot{K}_3 - 4(1+K_1)K_3 + 4\alpha K_1 - 2K_2^2]x^3 + \ldots + [\dot{K}_N - (N+1)(1+K_1)K_N - F_N(\alpha, K_1, \ldots, K_{N-1})]x^N = 0$$

Since the above equation must hold at every point in the state space (i. e. , for all $x$), we conclude that all [ ]'ed coefficients must vanish independently, this provides the following equations:

$$\begin{array}{lcl} \dot{K}_1 - 2K_1 - K_1^2 + 1 = 0 & \Longrightarrow & K_1 \\ \dot{K}_2 - 3(1+K_1)K_2 = -3\alpha K_1 & \Longrightarrow & K_2 \\ \dot{K}_3 - 4(1+K_1)K_3 = -4\alpha K_1 + 2K_2^2 & \Longrightarrow & K_3 \\ \vdots & & \\ \dot{K}_N - (N+1)(1+K_1)K_N = F_N(\alpha, K_1, K_2, \ldots, K_{N-1}) & \Longrightarrow & K_N \end{array} \tag{2.2.34}$$

Notice the following structure and properties of Eqs. (2.2.34) satisfied by the optimal gains:

- These equations can be solved *sequentially* for the $K_i$, all but the first are linear equations.
- The equations can solved to arbitrary order, since we have developed explicit recursions for $F_N$.
- The first equation for $K_1$, is a scalar Riccati equation (no surprise here!).
- If we impose $\alpha = 0$ and the boundary conditions $K_i(t_f) = 0$ [consistent with $\lambda(t_f) = 0$ as a transversality condition for $x(t_f)$ free], we see that all nonlinear gains vanish identically and therefore the above is indeed a direct generalization of the classical linear regulator with a quadratic performance index, the optimal control of Eq. (2.2.33) reduces to $u = -K_1 x$.
- If $t_f \to \infty$ , analogous to the classical steady state regulator, we can show that all $K_i$ approach constants and Eqs. (2.2.34) therefore reduce to a sequence of algebraic equations for the constant gains (we can show that the positive real root is the proper selection for $K_1$), and the solution for the higher gains involves only simple algebraic operations. For this case, with $\alpha = 1$, the first few gains are numerically:

$$K_1 = 0.41321,\quad K_2 = 0.29289,\quad K_3 = 0.17678,\quad K_4 = 0.088388,\quad K_5 = 0.033145.$$

Shown below in Figure 2.3 is the performance index versus the order $N$ of the feedback control and the trajectories of $x(t)$ and $u(t)$ for typical initial conditions [$x(0)=1.3$]. As is qualitatively evident, the nonlinear terms are constructive and convergence is rapid. We have shown that the nonlinear controls for N>3 are globally stable, whereas large but finite domains of stability are associated with the linear and quadratic feedback control laws. This solution has been verified by the symbolic manipulator code. As we indicate below, the structure of this simplest scalar example fully generalizes, so that the above observations apply in a much more general context. However, the stability and convergence issues are problem-dependent, as should be expected. We have established a means to generate the matrix equivalent of the sequence of scalar equations ( Eqs. (2.2.34) ) and their solutions for the optimal nonlinear feedback gains

**Solution of Problem II**

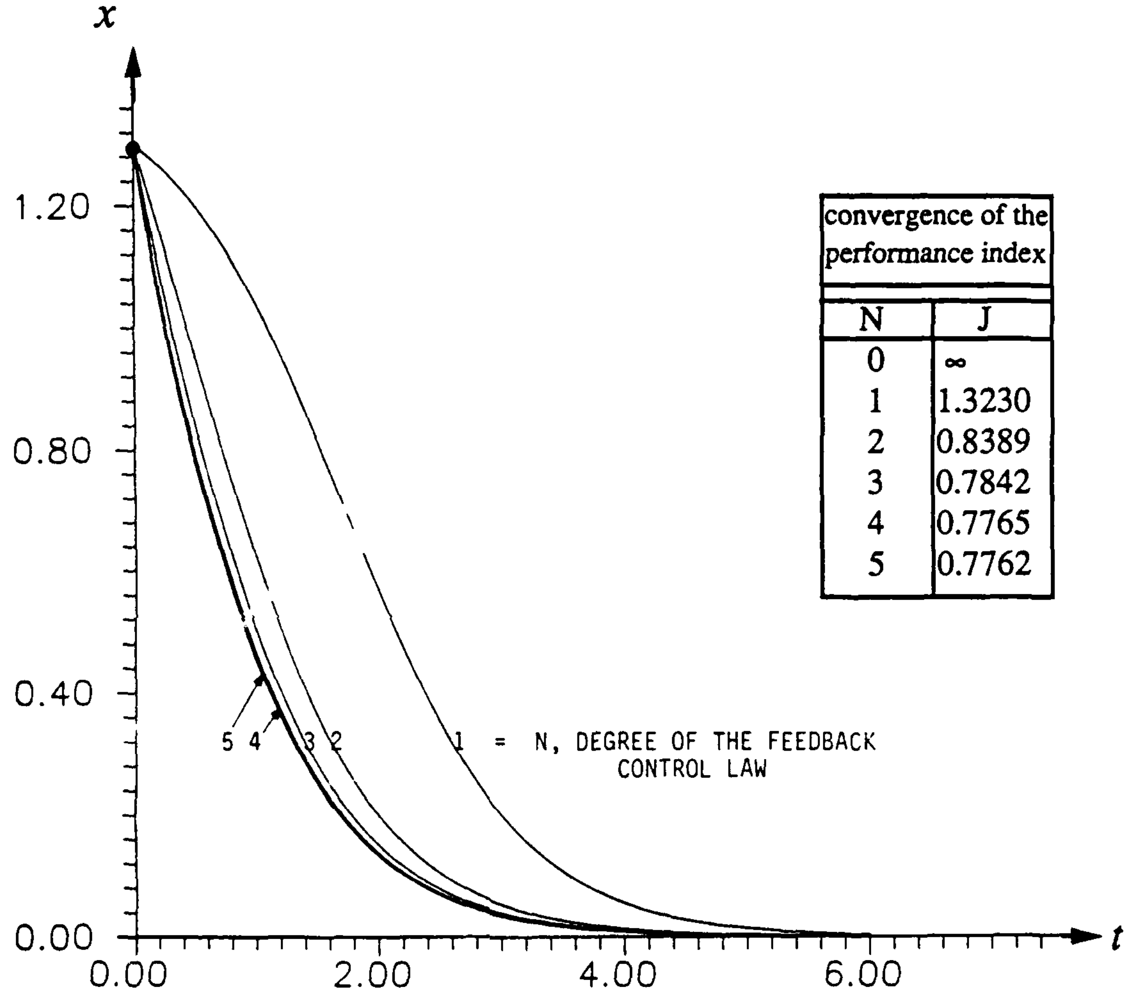
We seek an optimal feedback control law to compute $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ for the system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 + x_2^2 + u_1 \\ \dot{x}_2 &= -x_2 + x_1 x_2 + x_1^2 + u_2 \end{aligned} \tag{2.2.35}$$

which minimizes

$$J = \frac{1}{2}\int_0^{\infty} \sum_{i=1}^{2} (x_i^2 + u_i^2)\, dt = \frac{1}{2}\int_0^{\infty} (x^T Q x + u^T R u)\, dt \tag{2.2.36}$$

Eqs. (2.2.35) can be written as $\dot{x} = A_1 x_1 + A_2 x_2 + Bu$ , $\qquad Q = R = I \qquad$ (2.2.37)




| N | J |
|---|---|
| 0 | ∞ |
| 1 | 1.3230 |
| 2 | 0.8389 |
| 3 | 0.7842 |
| 4 | 0.7765 |
| 5 | 0.7762 |

**Figure 2.3 State Trajectory Versus Time : Effects of Increasing the Degree of Nonlinear Feedback** $(x0) = 1.3)$

where $x_1^T = x^T = [x_1 \;\; x_2]$, $x_2^T = [x_1^2 \;\; x_1 x_2 \;\; x_2^2]$, $u^T = [u_1 \;\; u_2]$, $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The Hamiltonian function is $H = \frac{1}{2}\left(x^T x + u^T u\right) + \lambda^T (A_1 x_1 + A_2 x_2 + Bu)$ (2.2.38)

The Pontryagin conditions are [5]

$$\left\{ \begin{array}{c} \dfrac{\partial H}{\partial u} = 0 \;\Rightarrow\; u = -B^T \lambda \\ \dot{\lambda} = -\dfrac{\partial H}{\partial x} = -x - A_1^T \lambda - \left[\dfrac{\partial x_2^T}{\partial x_1}\right] A_2^T \lambda \\ \dot{x} = \dfrac{\partial H}{\partial \lambda} = A_1 x_1 + A_2 x_2 - B B^T \lambda \\ \text{where} \left[\dfrac{\partial x_2^T}{\partial x_1}\right] = \begin{bmatrix} 2x_1 & x_2 & 0 \\ 0 & x_1 & 2x_2 \end{bmatrix} \end{array} \right\} \quad (2.2.39)$$

We seek to determine the feedback gains $K_i$ in the control law expansion

$$u = -B^T \lambda = \sum_{i=1}^{N} G_i x_i, \text{ where } \lambda = \sum_{i=1}^{N} K_i x_i \;,\; G_i = -B^T K_i \quad (2.2.40)$$

Substitution of the expansion of Eqs. (23) into the necessary conditions of Eq. (24) and collecting terms leads directly to a system of two homogeneous conditions of the following form

$$F_{1i}(K_1)x_1 + F_{2i}(K_1)x_2 + F_{11i}(K_1, K_2)x_1^2 + F_{12i}(K_1, K_2)x_1 x_2 + F_{22i}(K_1, K_2)x_2^2 + \ldots = 0 \,,\, i = 1, 2$$

Requiring these conditions to hold at every point in the state space, we can set the coefficients of all powers and products of the elements of $x$ to zero. Upon carrying through the algebra, we find the linear terms yield four algebraic equations $F_{11}(K_1) = F_{12}(K_1) = F_{21}(K_1) = F_{22}(K_1) = 0$, which are precisely the four elements of the Riccati equation [29, 30]

$$K_1 A_1 + A_1^T K_1 - K_1 B R^{-1} B^T K_1 + Q = 0 \quad (2.2.41)$$

The Riccati equation can be solved for the symmetric linear gain matrix $K_1$. Setting the six quadratic term's coefficients to zero yields $F_{11i}(K_1, K_2) = F_{12i}(K_1, K_2) = F_{22i}(K_1, K_2) = 0$, $i = 1, 2$; we find that these six algebraic equations are linear in the six distinct elements of $K_2$, and can be brought to the form of the linear system

$$[\mathbf{L}_2(K_1)]\, \mathrm{vec}\{K_2\} = \mathbf{R}_2(K_1) \quad (2.2.42)$$

where $[\mathbf{L}_2(K_1)]$ is a 6x6 matrix whose elements are functions of $K_1$, $\mathbf{R}_2(K_1)$ is a 6x1 vector whose elements depend upon $K_1$, and $\mathrm{vec}\{K_2\}$ is a 6x1 vector whose elements *are the six distinct elements of* $K_2$. Obviously, if $[\mathbf{L}_2(K_1)]$ is of full rank, Eq. (2.2.42) can be inverted for $\mathrm{vec}\{K_2\}$.

To conserve space, we do not write out the algebraic equations for the elements of these matrices. We find that Eq. (2.2.42) generalizes; the higher order gains are determined by a *sequence* of linear equations of the form

$$[\mathbf{L}_k(K_1, K_2, ..., K_{k-1})]\,\mathrm{vec}\{K_k\} = \mathbf{R}_k(K_1, K_2, ..., K_{k-1}),\ k = 2, 3, \ldots N \qquad (2.2.43)$$

For the case of particular $A_i$, $B$, of Eqs. (2.2.20), and $Q = R = I$, we have carried through the above developments and find the following numerical values for the first five control gains ($G_i = -R^T K_i$):

$$G_1 = -\begin{bmatrix} 0.41421 & 0 \\ 0 & 0.41421 \end{bmatrix},\quad G_2 = -\begin{bmatrix} 0 & 0.39052 & 0.19526 \\ 0.19526 & 0.39052 & 0 \end{bmatrix},\quad G_3 = -\begin{bmatrix} 0.12459 & 0.27022 & 0.22222 & 0.09007 \\ 0.09007 & 0.22222 & 0.27022 & 0.12459 \end{bmatrix}$$

$$G_4 = -\begin{bmatrix} 0.05125 & 0.17517 & 0.26213 & 0.17475 & 0.04379 \\ 0.04379 & 0.17475 & 0.26213 & 0.17517 & 0.05125 \end{bmatrix},\quad G_5 = -\begin{bmatrix} 0.01656 & 0.08537 & 0.16605 & 0.16225 & 0.08302 & 0.01707 \\ 0.01707 & 0.08302 & 0.16225 & 0.16605 & 0.08537 & 0.01656 \end{bmatrix}$$

Since this system is highly nonlinear, simply ignoring the nonlinear terms and deriving an approximate linear optimal control law will be valid only near the origin. In Figure 2.4 we show graphs of the stable region (shaded) vs $N$ and typical trajectories of the state and control variables for typical initial conditions, for controllers based upon linear ($N$=1) through quintic ($N$=5) feedback. Notice (as might be anticipated), the linear feedback law does not globally stabilize this nonlinear system, but it does near the origin of the state space. Including the quadratic terms modifies the stable region, increasing the stable domain area in the positive $(x_1, x_2)$ quadrant, but decreasing it elsewhere. Including the third degree terms results in *global* stability.

There is evidence that outside the shaded region of Figure 2.4b, all even degree feedback controllers are unstable, whereas inside this region *all* controllers with 2≤N≤5 are stable and converge rapidly to the optimal control. Using Lyapunov methods, we determined an N=2 globally stabilizing, sub-optimal quadratic feedback law for this system, but this does not detract from the significance of the above results, since they generalize fully to the case of higher order systems with a small number of controllers, in which case no general method exists to construct a control law for which global stability is guaranteed. Indeed, we believe the ideas we are discussing can be developed into a widely applicable method for enlarging the stable, near-optimally controlled region for a large class of nonlinear systems. It is evident that including the nonlinear terms in the truncated feedback control law is constructive and convergence to near optimal solutions can be achieved over greatly enlarged regions of the state space. However, careful evaluation of the solution behavior is still required, there are no guarantees of monotonic convergence.

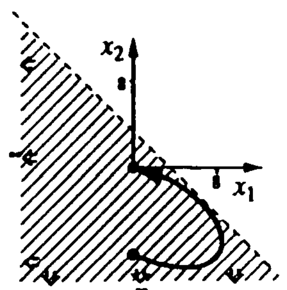



Figure 2.4a Stable Region for Linear Feedback Law

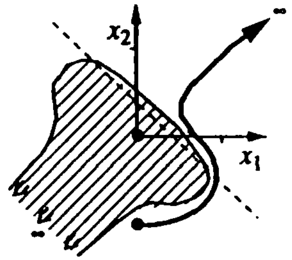



Figure 2.4b Stable Region for Quadratic Feedback Control Law

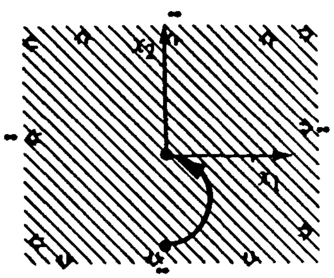



Figure 2.4c Cubic Feedback Results in Global Stability

Phase Portraits: Stable Regions for Linear Quadratic, and Cubic Feedback Control

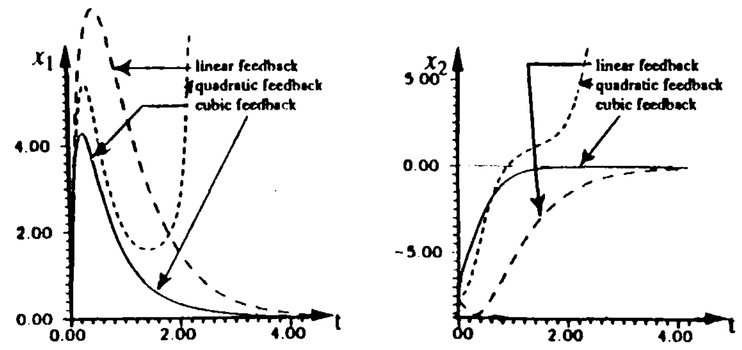



Figure 2.4d State Trajectory Versus Time Starting From (0,-8)

convergence of the performance index

| N | J |
|---|---|
| 1 | 68.5 |
| 2 | ∞ |
| 3 | 63.3 |
| 4 | ∞ |

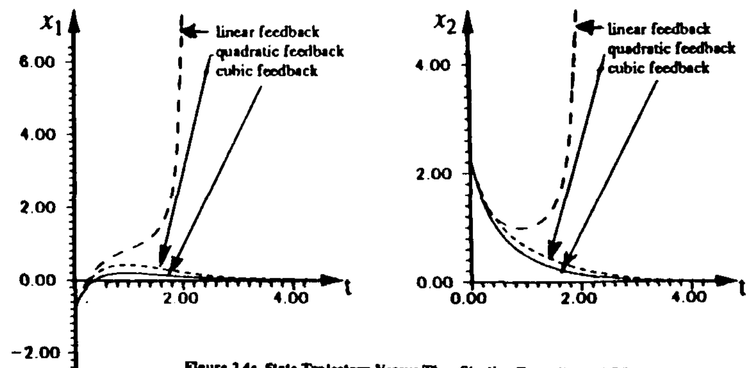



Figure 2.4e State Trajectory Versus Time Starting From (2.25, -0.75)

convergence of the performance index

| N | J |
|---|---|
| 1 | ∞ |
| 2 | 1.35 |
| 3 | 1.09 |
| 4 | 1.08 |
| 5 | 1.08 |

Trajectories Versus Time and Convergence of the Performance Index Versus Degree of Nonlinear Feedback

**Figure 2.4 Stability Maps and Typical Trajectories for Problem II: Effects of Increasing the Degree of the Nonlinear Feedback**

**Solution of Problem III** (Optimal Nonlinear Feedback Control of Spacecraft Attitude)

We seek an optimal feedback control law to compute the feedback torque $\boldsymbol{u(x)}$ to maneuver the system modeled by Eqs. (2.2.23) or more generally Eqs. (2.2.25) to the origin of the state space ($\boldsymbol{x} = \boldsymbol{0}$) in such a fashion that minimizes the index of Eq. (2.2.24). The detailed discussion of this example is .1ot given, but is analogous to the solution of Problem II. The expansions were carried out to 3rd order via the symbolic manipulator. This is essentially the same system considered in [29,30]. Here we present only a summary of the numerical solution of the resulting general symbolic gain equations for the particular system parameters: $I_1 = 1.00$, $I_2 = 0.83$, $I_3 = 0.92$, $Q$ , $R$ identity matrices, and unit initial conditions on all state variables.

For the first two gain matrices ($G_i = -R^{-1}B^TK_i$ ), where the $K_i$ are from the expansion of Eq. (2.2.28), we find the following numerical values

$$G_1 = -\begin{bmatrix} 1.000 & 0 & 0 & 1.414 & 0 & 0 \\ 0 & 1.000 & 0 & 0 & 1.353 & 0 \\ 0 & 0 & 1.000 & 0 & 0 & 1.386 \end{bmatrix},$$

$$G_2 = -10^{-5}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3184 & 0 & 0 & -978 & 0 & 0 & 1893 & 0 & 0 & 0 & 0 & 0 & -.2078 & 0 \\ 0 & 0 & -2764 & 0 & 0 & -1118 & 0 & 0 & 0 & 0 & 0 & 0 & 2280 & 0 & 0 & 0 & 0 & -.2504 & 0 & 0 & 0 \\ 0 & 5948 & 0 & 0 & -1009 & 0 & 0 & 0 & -1063 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.2259 & 0 & 0 & 0 & 0 \end{bmatrix}$$

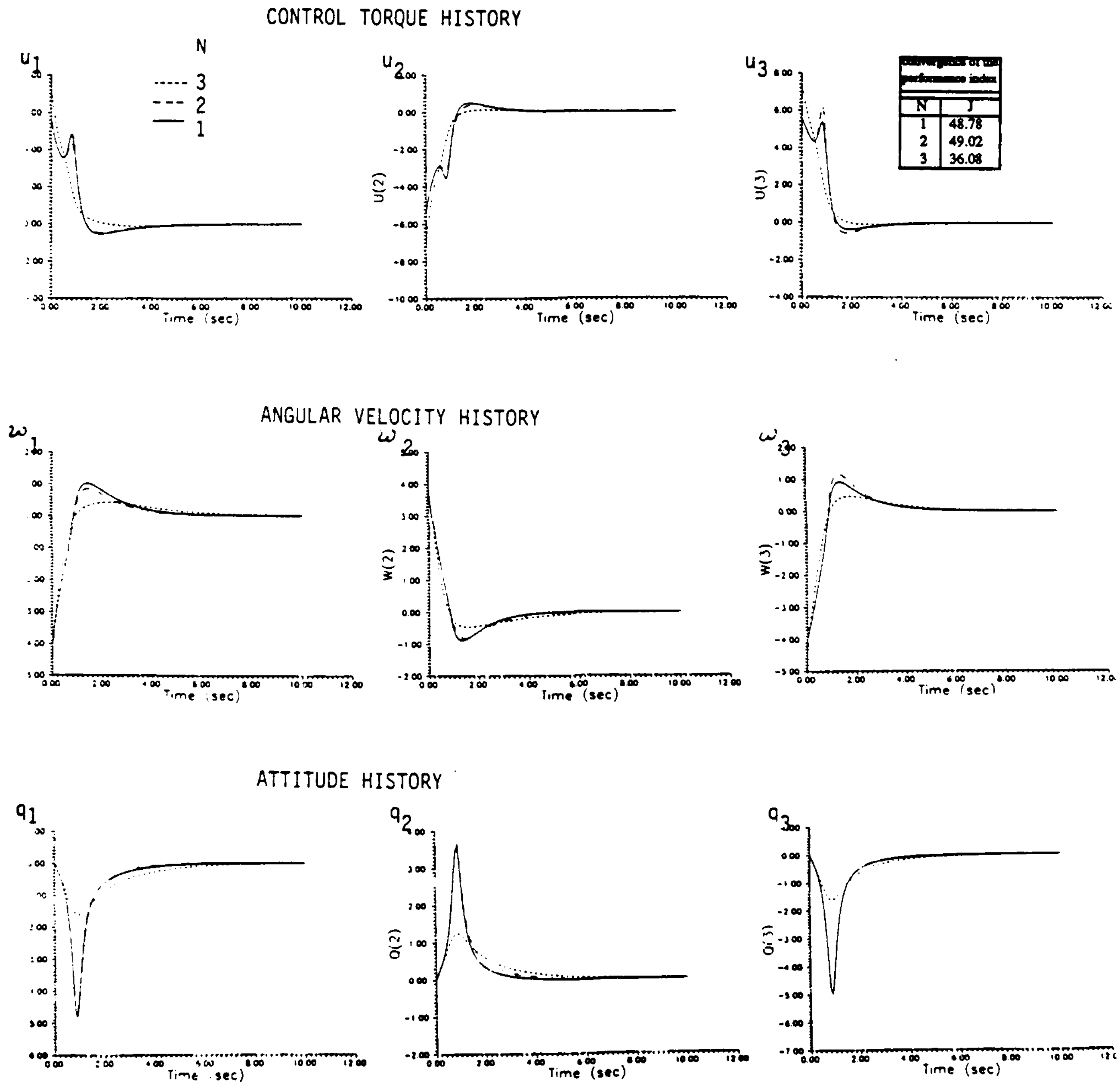
Due to space limitations, we show only a few digits and do not display the third order gain matrix which was also computed. As is apparent, the particular system dynamics, inertias, and identity weight matrices resulted in sparse gain matrices, this pattern carries over to the cubic gains. This sparsity pattern is unaffected by variations of the elements in the diagonal inertia, Q, and R matrices, although, obviously, the numerical values of the gains are affected. The sparsity pattern of the optimal gain matrices allows us to write down the form of the feedback control law explicitly, as follows:

$$\begin{aligned} u_1 &= G_1(1,1)q_1 + G_1(1,4)\omega_1 + G_2(1,8)q_2 q_3 + G_2(1,11)q_2\omega_3 + G_2(1,14)q_3\omega_2 + G_2(1,20)\omega_2\omega_3 + \ldots \\ u_2 &= G_1(2,2)q_2 + G_1(2,5)\omega_2 + G_2(2,3)q_3 q_1 + G_2(2,13)q_3\omega_1 + G_2(2,6)q_1\omega_3 + G_2(2,18)\omega_3\omega_1 + \ldots \\ u_3 &= G_1(3,3)q_3 + G_1(3,6)\omega_3 + G_2(3,2)q_1 q_2 + G_2(3,5)q_1\omega_2 + G_2(3,9)q_2\omega_1 + G_2(3,17)\omega_1\omega_2 + \ldots \end{aligned}$$

*linear feedback terms* *quadratic (note "gyroscopic" structure) feedback terms* *3rd & HOT*

While it is easy to conjecture the uncoupled structure of the linear terms in the optimal control (the linearized dynamical equations are uncoupled), and after-the-fact, the "gyroscopic" structure of the optimal quadratic feedback is intuitively reasonable, it is more difficult to anticipate the structure of the 3rd and higher order terms of the optimal control law. The third degree feedback gains are similarly sparse (only 15 of the 56 elements in each row of $G_3$ are non-zero), but are not displayed, for brevity. The beauty of the above method is that not only the structure of the control law, but also the optimal values of the gains can be routinely determined. Of course, fully populated weight and inertia matrices result in more densely populated gain matrices. Shown in Figure 2.5 are graphs of the the state and control variable trajectories corresponding to unit initial conditions (note the initial conditions correspond to large angular velocities about

**Figure 2.5 Perturbation Feedback Controlled Spacecraft Attitude Maneuver**



each axis -- thus the gyroscopic nonlinear effects are fairly pronounced). As is evident in Figure 2.5, we again find very constructive effects, upon augmenting the classical linear control gains with the optimal quadratic and cubic feedback, it is also obvious that rapid convergence is being achieved. Even though the quadratic and cubic control contributions are significant, we anticipate the fourth and higher order feedback contributions would make negligible contributions.

These results and those obtained in solving the first two examples appear to be typical, the absence of formal convergence proofs does not usually prevent us from obtaining practical solutions which do in fact display convincing evidence of convergence with low degree nonlinear feedback. Especially important is the promise of this approach to make the determination and revision of nonlinear, near-optimal controls relatively straight forward, so that convergence can be studied and we can efficiently incorporate system and performance index modifications.

### 2.3.3 Status and Outlook

As is evident from the above developments, we have made some significant progress in the development and preliminary evaluation of a new approach to nonlinear feedback control. We have however encountered several dimensionality - related difficulties. The number of symbolic operations depends upon the second power of the product MN (M = highest degree of polynomial nonlinearity, N = order of the dynamical system); we have been successful in carrying to completion the design for several examples for which $MN < 20$. However, it presently appears that our present formulation and implementation will not be practical if MN exceeds about 40 due to excessive computational demands. However, we are optimistic that new developments which adaptively exploit sparsity structure associated with each system (instead of the present most general approach which initially allows for all nonlinear terms of degree M and lower) will be developed which will permit higher dimensioned applications.

A more serious difficulty, associated with the lack of stability guarantees, has been encountered. We have found several examples in which nonlinear controllers of certain degrees were unstable; this was found after the fact (after the optimality conditions were applied and the gains computed). Unlike linear systems (for a linear controllable system, it can be proven that minimizing a quadratic index always leads to a stable controller), there is no proof that polynomial (truncated at some degree) feedback, determined to minimize some positive performance index will stabilize a given nonlinear system. Indeed we have encountered several counter-examples. These issues will be studied further in the next year.

## 3.0 Concluding Remarks

Section 2 summarizes progress we have made on three sets of research problems:

> Penalty methods for simulation of flexible multibody dynamics.
>
> A power method for design of output feedback controllers for nonlinear distributed parameter systems
>
> A symbolic approach for designing full state feedback control laws for dynamical systems with polynomial nonlinearities. polynomial

On all three sets of problems, we have obtained some fundamental analytical results and have studied prototype applications to evaluate salient features and the practical potential of the methodology. In all three areas substantial progress has been made as reported above, and as discussed in the sub-sections 2.1.4, 2.2.3, and 2.3.3, we are aggressively pursuing extensions to the results in all three areas of investigation.

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# Attachment Index

1. Globally Stable Feedback Controllers for Nonlinear Distributed Parameter Systems

2. A Nonrecursive "Order N" Preconditioned Conjugate Gradient/Range Space Formulation of MDOF Dynamics

3. Time Domain convergence Properties of Lyapunov Stable Penalty Methods

ATTACHMENT 1

# Globally Stable Feedback Controllers for Nonlinear distributed Parameter Systems

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*8th Annual Forum on Space Structures*
*Melbourne, Florida*
*June 18-20, 1990*

# Globally Stable Feedback Controllers for Nonlinear Distributed Parameter Systems

*John L. Junkins*
Texas A&M University

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# OUTLINE

**Fondest Hopes, Wildest Dreams . . .** Mark Balas Lives!

**Example One:** Fine Pointing/Vibration Suppression
*globally stable control law for a nine body DPS system.*

**Example Two:** Large Angle Maneuver: Near-Min-Time Control
*globally stable tracking control for DPS of Example One.*
analytical results ... numerical results ... experimental results

**Generalization of the Methodology**
A *Power Method* for designing stable feedback controllers of nonlinear, multi-body systems modeled by ODE/PDEs.

**Where To From Here?**

## *Fondest Hopes and Wildest Dreams: A Wish List*

**It "would be nice" if a robust control design methodology existed with the following attributes:**

- **The method readily accommodates *Nonlinearity***

- **The method applies directly to *hybrid systems of ODES & PDEs* ... with no spatial discretization, truncation, or spillover errors in the stability proof!**

- **In the absence of disturbances, the method leads to *Global Stability*, if the system is controllable.**

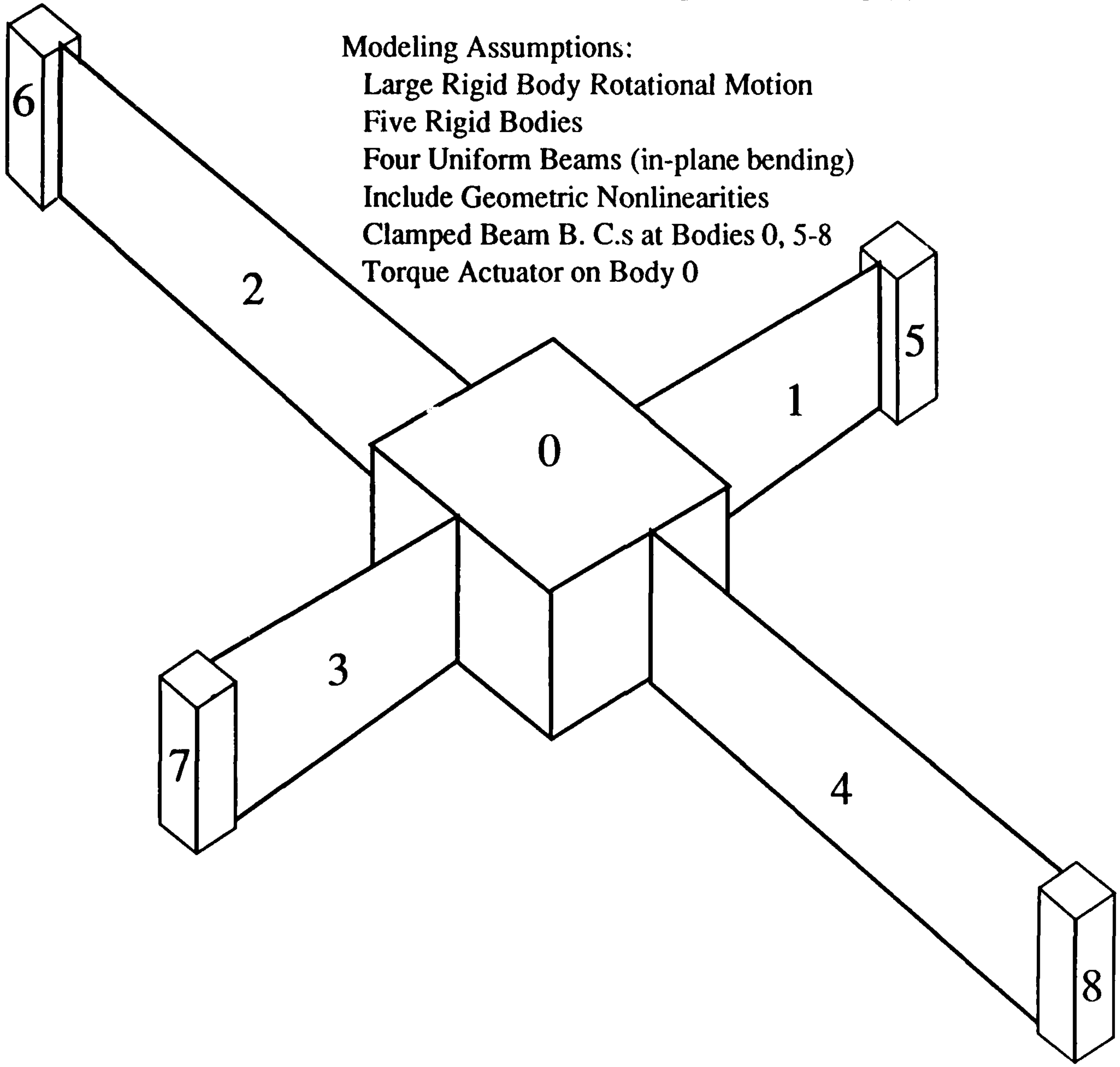
- **The method can accommodate tradeoffs between *competing measures of optimality*, for example:**

  ***minimum time* vs *minimum control energy* vs *small performance errors***

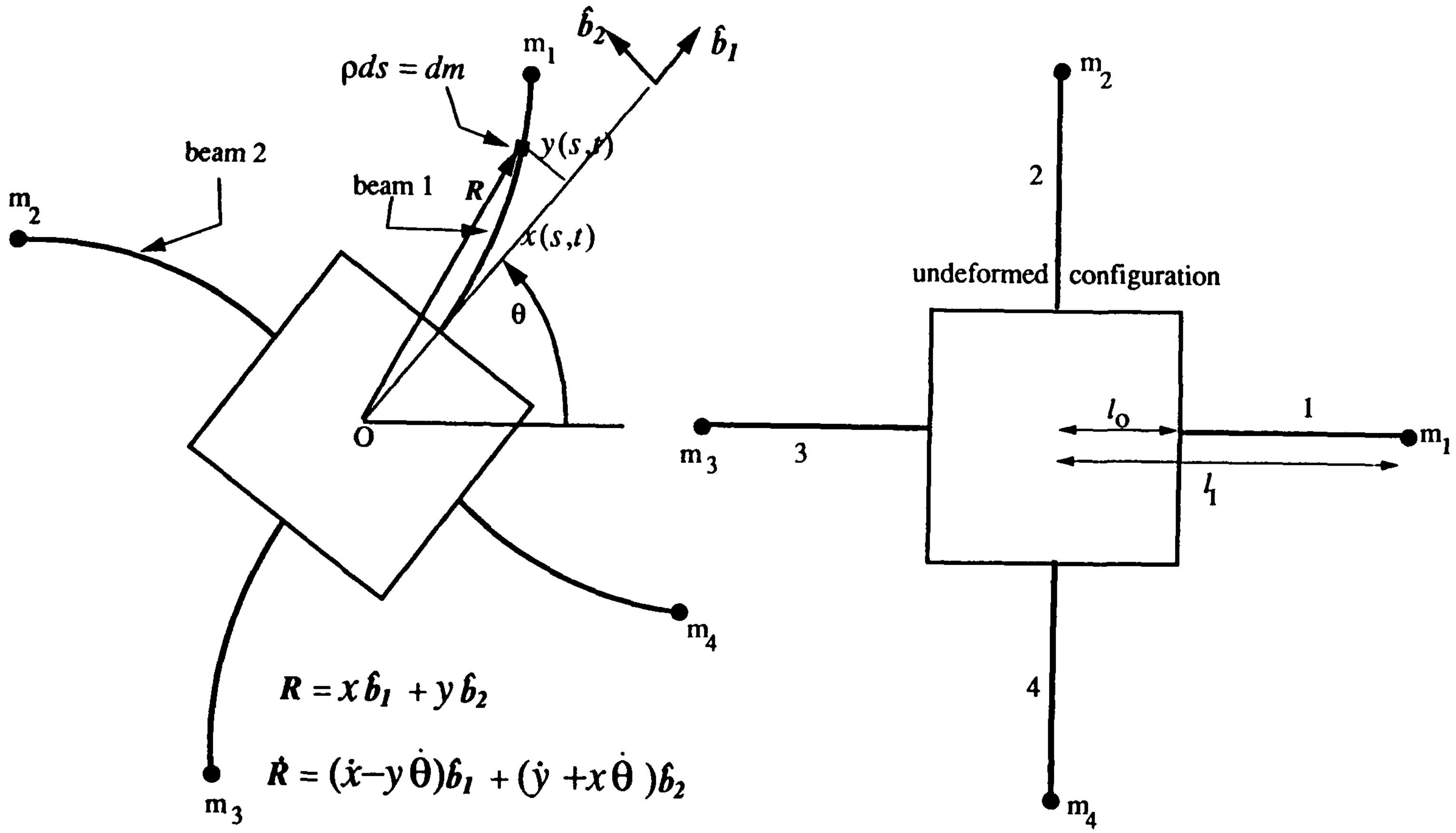
**Well I have some good news, *the method discussed below has these attributes.***

# A NINE BODY CONFIGURATION

Modeling Assumptions:
- Large Rigid Body Rotational Motion
- Five Rigid Bodies
- Four Uniform Beams (in-plane bending)
- Include Geometric Nonlinearities
- Clamped Beam B. C.s at Bodies 0, 5-8
- Torque Actuator on Body 0



# Toward the System Equations of Motion . . .




$$\mathbf{R} = x\hat{b}_1 + y\hat{b}_2$$

$$\dot{\mathbf{R}} = (\dot{x} - y\dot{\theta})\hat{b}_1 + (\dot{y} + x\dot{\theta})\hat{b}_2$$

<u>System Lagrangian:</u>

$$L = T - V = T_{hub} + T_{beam} + T_{tip} - V_{beam} = \tfrac{1}{2} I_{hub}\dot{\theta}^2 + \sum_{i=1}^{4}\Big[\tfrac{1}{2}\int_{l_o}^{l_i} \dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\rho_i\, ds_i + \tfrac{1}{2} m_i \dot{\mathbf{R}}_{tip_i}\cdot\dot{\mathbf{R}}_{tip_i} - \tfrac{1}{2}\int_{l_o}^{l_i} EI_i\left(\frac{\partial^2 y_i}{\partial s_i^2}\right)^2 ds_i\Big] + \ldots$$

*Apply the Extended Hamilton's Principle:* $\int_{t_1}^{t_2}(\delta L + \delta W)dt \; + \; BCs \Rightarrow$ *PDE equations of motion* $\Rightarrow$

# EQUATIONS OF MOTION

The resulting hybrid system of equations of motion are

$$I_{hub}\frac{d^2\theta}{dt^2} = u + \sum_{i=1}^{4}(\mathrm{M_o} - \mathrm{S_o}\, l_o)_i$$

$$-(\mathrm{M_o} - \mathrm{S_o}\, l_o)_i = \int_{l_o}^{l_i} \rho_i x_i \left(\frac{\partial^2 y_i}{\partial t^2} + x_i \frac{d^2\theta}{dt^2}\right) dx_i \; + \; m_i\left(l_i \frac{d^2\theta}{dt^2} + \frac{\partial^2 y_i}{\partial t^2}\Big|_{l_i}\right) + HOT \tag{1}$$

$$\rho_i\left(\frac{\partial^2 y_i}{\partial t^2} + x_i \frac{d^2\theta}{dt^2}\right) + \mathrm{EI}\frac{\partial^4 y_i}{\partial x_i^4} = 0 + HOT \; , \quad i = 1, 2, 3, 4$$

Higher order terms (*HOT*) indicate other *known* linear & nonlinear effects (such as rotational inertia effects, rotational stiffening, foreshortening effects, shear deformation, etc.). The most fundamental of the developments presented below do not consider these higher order effects, however, we have extended the results to consider these effects as well. The B. C.s on Eqs. (1) are

$$at\ x_i = l_o:\ y_i(t, l_o) = 0, \qquad \frac{\partial y_i}{\partial x}\Big|_{l_o} = 0$$

$$at\ x_i = l_i:\ \frac{\partial^2 y_i}{\partial x_i^2}\Big|_{l_i} = 0 + HOT, \qquad \frac{\partial^3 y_i}{\partial x_i^3}\Big|_{l_i} = \frac{m_i}{\mathrm{EI}_i}\left(l\frac{d^2\theta}{dt^2} + \frac{\partial^2 y_i}{\partial t^2}\Big|_{l_i}\right) \tag{2}$$

# SYSTEM ENERGY & JUDICIOUS LIAPUNOV FCTS.

The total energy of the system (constant in the absence of control or disturbances) is:

$$2E = I_{hub}\left(\frac{d\theta}{dt}\right)^2 + \sum_{i=1}^{4}\left[\int_{l_o}^{l_i}\rho_i\left(\frac{\partial y_i}{\partial t} + x_i\frac{d\theta}{dt}\right)^2 dx_i + \int_{l_o}^{l_i} EI_i\left(\frac{\partial^2 y_i}{\partial x_i^2}\right)^2 dx_i + m_i\left(l_i\frac{d\theta}{dt} + \frac{\partial y_i}{\partial t}\Big|_{l_i}\right)^2\right] + HOT \quad (3)$$

In view of the energy integral (which is a constant of the motion for zero control), we investigate the Liapunov function

$$2U = a_0 I_{hub}\left(\frac{d\theta}{dt}\right)^2 + \sum_{i=1}^{4}a_i\left[\int_{l_o}^{l_i}\rho_i\left(\frac{\partial y_i}{\partial t} + x_i\frac{d\theta}{dt}\right)^2 dx_i + \int_{l_o}^{l_i} EI_i\left(\frac{\partial^2 y_i}{\partial x_i^2}\right)^2 dx_i + m_i\left(l_i\frac{d\theta}{dt} + \frac{\partial y_i}{\partial t}\Big|_{l_i}\right)^2\right] + HOT$$
$$+ a_5(\theta - \theta_f)^2 \quad (4)$$

The positive weighting coefficients $a_i > 0$ are introduced to allow relative emphasis upon five substructures' contributions to the total *error energy* of the system. Note that the open loop system energy integral of Eq. (3) does not depend upon the rigid body displacement, *thus the final term is introduced subject to the constraint that the functional of Eq. (4) has its global minimum at the target final state:*

$$\{\theta, \dot\theta\}_{desired} = (\theta_f, 0), \qquad \left\{ y_i(x_i, t), \frac{\partial y_i(x_i, t)}{\partial t}\right\}_{desired} = (0, 0), \quad i=1,2,3,4$$

More generally, error energy can be measured from a time varying *target trajectory*.

# Dissipation of Error Energy ⇒ Liapunov Stable Control Laws

Recall our candidate Liapunov function is the error energy functional:

$$2U = a_0 I_{hub}\dot{\theta}^2 + \sum_{i=1}^{4} a_i \left[ \int_{l_o}^{l_i} \rho_i \left(\frac{\partial y_i}{\partial t} + x_i\dot{\theta}\right)^2 dx_i + \int_{l_o}^{l_i} EI_i \left(\frac{\partial^2 y_i}{\partial x_i^2}\right)^2 dx_i + m_i \left(l_i\dot{\theta} + \frac{\partial y_i}{\partial t}\Big|_{l_i}\right)^2 \right] + a_5(\theta-\theta_f)^2 \quad (4)$$

Differentiation of Eq. (4), substitution of the equations of motion (Eqs. (1), (2)), and some calculus leads to

$$\dot{U} = \frac{dU}{dt} = \dot{\theta}\left[ a_0 u + a_5(\theta-\theta_f) + \sum_{i=1}^{4}(a_i - a_0)(l_o S_o - M_o)_i \right] \equiv \mathit{weighted}\ \underline{\mathit{power}} \quad (5)$$

Since we require that $\dot{U} \le 0$, we set the [ ] term to $-a_6\dot{\theta}$ and this leads to $\dot{U} = -a_6\dot{\theta}^2$ and

the control law:
$$u = -\frac{1}{a_0}\left[ a_5(\theta-\theta_f) + a_6\dot{\theta} + \sum_{i=1}^{4}(a_i - a_0)(l_o S_o - M_o)_i \right] \quad (6)$$

Thus we see that the following *linear, spatially discrete* output feedback law satisfies the *sufficient condition* ($\dot{U} \le 0$) to globally stabilize this distributed parameter system:

$$u = -\left[ g_1(\theta-\theta_f) + g_2\dot{\theta} + \sum_{i=3}^{6} g_i(l_o S_o - M_o)_i \right],\ g_1 \equiv \frac{a_5}{a_0} \ge 0,\ g_2 \equiv \frac{a_6}{a_0} \ge 0,\ g_i \equiv \frac{a_{i-2} - a_0}{a_0} \ge -1 \quad (7)$$

The *pervasive dissipation condition that* $\dot{U} = -a_6\dot{\theta}^2$ *is strictly negative*, for asymptotic stability, is satisfied *only if the system is fully controllable*. In the linear case, we find that the *anti-symmetric in opposition* modes, (for a perfectly symmetric structure, 4 identical appendages) have zero hub motion & are uncontrollable by a hub torque actuator.

# Stability Robustness: ⇒ Toward a General Methodology

It is significant that *if one includes* *all of the following linear and nonlinear HOT effects:*

shear deformation, rotary inertia,
foreshortening effects, rotational stiffening, . . . .,

and *consistently* modifies the equations of motion, total energy, and writes the functional:

$$2U = a_0 I_{hub} \dot{\theta}^2 + \sum_{i=1}^{4} a_i \left[\text{beam K. E.} + \text{beam P. E.} + \text{tip mass K. E.}\right]_i + a_5 (\theta - \theta_f)^2 \qquad (4)'$$

Then, differentiation of Eq. (4)' substitution of the appropriately modified equations of motion (Eqs. (1)' (2)'), and some calculus leads to the identical result

$$\dot{U} = \frac{dU}{dt} = \dot{\theta}\left[ a_0 u + a_5 (\theta - \theta_f) + \sum_{i=1}^{4} (a_i - a_0)(l_o S_o - M_o)_i \right] \equiv \textit{weighted } \underline{\textit{power}} \qquad (5)$$

Thus requiring that $\dot{U} \le 0$ by setting the [ ] term to $-a_6 \dot{\theta}$ giving $\dot{U} = -a_6 \dot{\theta}^2$ again leads to the control law:

$$u = -\left[ g_1 (\theta - \theta_f) + g_2 \dot{\theta} + \sum_{i=3}^{6} g_i (l_o S_o - M_o)_i \right], \; g_1 \equiv \frac{a_5}{a_0} \ge 0, \; g_2 \equiv \frac{a_6}{a_0} \ge 0, \; g_i \equiv \frac{a_{i-2} - a_0}{a_0} \ge -1 \qquad (7)$$

The invariance of this law (w.r.t. model assumptions) is as elegant as it is fundamental. It generalizes local velocity feedback, note that shear and bending are measurable via strain gauges). However, similar to local velocity feedback, *stability robustness is obtained for all passive structural models.* Controllability must be analyzed on a case by case basis & optimization of the $g_i$ (over the stable region) & depends upon the system model.

# Work/Energy Principle $\Rightarrow$ Power = energy rate (due to working forces)

**Energy Equation Form Depends upon the System (*or Sub System*) Model**

| System Model | Kinetic Energy Form | Work/Energy Eqn. | Power Eqn. |
|---|---|---|---|
| system of particles | $T \equiv \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\mathbf{R}}_i \bullet \dot{\mathbf{R}}_i$ | $T - T_o = \int_{t_o}^{t} \sum_{i=1}^{N} \mathbf{F}_i \bullet \dot{\mathbf{R}}_i \, dt$ | $\frac{dT}{dt} = \sum_{i=1}^{N} \mathbf{F}_i \bullet \dot{\mathbf{R}}_i$ |
| continuum | $T \equiv \frac{1}{2}\int_v \dot{\mathbf{R}} \bullet \dot{\mathbf{R}} \rho dv$ | $T - T_o = \int_{t_o}^{t} \int_v \mathbf{F} \bullet \dot{\mathbf{R}} \, dv dt$ | $\frac{dT}{dt} = \int_v \mathbf{F} \bullet \dot{\mathbf{R}} \, dv$ |
| rigid body | $T \equiv \frac{1}{2} M \dot{\mathbf{R}}_p \bullet \dot{\mathbf{R}}_p + \frac{1}{2} I_p \omega_p \bullet \omega_p$ | $T - T_o = \int_{t_o}^{t} \mathbf{F} \bullet \dot{\mathbf{R}}_p \, dt + \int_{t_o}^{t} \mathbf{L} \bullet \omega_p \, dt$ | $\frac{dT}{dt} = \mathbf{F} \bullet \dot{\mathbf{R}}_p + \mathbf{L} \bullet \omega_p$ |

When one introduces the decomposition of forces $\mathbf{F} = -\nabla V + \mathbf{F}_{nw} + \mathbf{F}_w$, then all conservative forces can be absorbed into a change in potential energy; the above table becomes

| System | Work/Energy Equation | Power Equation |
|---|---|---|
| system of particles | $E = T + V = T_o + V_o + \sum_{i=1}^{N} \mathbf{F}_{wi} \bullet \dot{\mathbf{R}}_i \, dt$ | $\frac{dE}{dt} = \sum_{i=1}^{N} \mathbf{F}_{wi} \bullet \dot{\mathbf{R}}_i$ |
| continuum | $E = T + V = T_o + V_o + \int_{t_o}^{t} \int_v \mathbf{F}_w \bullet \dot{\mathbf{R}} \, dv dt$ | $\frac{dE}{dt} = \int_v \mathbf{F}_w \bullet \dot{\mathbf{R}} \, dv$ |
| rigid body | $E = T + V = T_o + V_o + \int_{t_o}^{t} \mathbf{F}_w \bullet \dot{\mathbf{R}}_p \, dt + \int_{t_o}^{t} \mathbf{L}_w \bullet \omega_p \, dt$ | $\frac{dE}{dt} = \mathbf{F}_w \bullet \dot{\mathbf{R}}_p + \mathbf{L}_w \bullet \omega_p$ |

# Weighted Power Decomposition by Substructures

## *Energy Liapunov Functions ⇒ Stable Control Laws*

Let the candidate Liapunov function be formed as a weighted linear combination of the total energies $E_i$ of $N+1$ substructures

$$2U = \sum_{i=0}^{N} a_i E_i + a_{N+1} f(q_1, q_2, \ldots) \tag{8}$$

where $f(q_1, q_2, \ldots)$ is selected to eliminate rigid body degrees of freedom and make $U$ a positive definite error energy measure with its global minimum at some prescribed target state. Differentiation of Eq. (8) gives

$$\frac{dU}{dt} = \sum_{i=0}^{N} a_i \frac{dE_i}{dt} + a_{N+1} \{ \frac{\partial f}{\partial q_1} \dot{q}_1 + \frac{\partial f}{\partial q_2} \dot{q}_2 + \ldots \} \tag{9}$$

Introducing a further substructure decomposition of the working forces (& similar result for moments) $\mathbf{F}_{wj} = \Sigma \mathbf{F}_{\mathbf{boundary}j} + \Sigma_{k=1}^{m} b_{jk} u_k = \textit{boundary forces} + \textit{control forces}$ (10)
& using the particular power equation for each substructure, Eq. (9) reduces to the form

$$\dot{U} = \Sigma_k^m \, h_k(u_1, u_2, \ldots, u_m; a_0, a_1, \ldots; q_1, q_2, \ldots; \mathbf{F}_{\mathbf{boundary}_1}, \mathbf{F}_{\mathbf{boundary}_2}, \ldots)\dot{q}_k \tag{11}$$

For stability, we can set $h_k(.)$ equal to *any* function having the opposite sign of $\dot{q}_k$, and, in principle, solve the resulting algebraic equations for the form of the control law for $u_i(.)$.

For the hub/appendage example: $\dot{U} = [\, a_0 u + a_5 (\theta - \theta_f) + \sum_{i=1}^{4} (a_i - a_0)(l_o S_o - M_o)_i \,]\dot{\theta}$ . (5)

## Near-Minimum-Time Maneuvers of DPS: Qualitative Issues

For the case of a rigid body, strict minimum time control is a bang-bang law.

> For the rest-to-rest maneuver-to-the-origin case, the control saturates negatively during the first half of the maneuver and positively during the last half of the maneuver, with all control switches occurring instantaneously.

From an implementation point of view, the instantaneous switches of bang-bang control are often troublesome for several reasons:

(i) No torque-generating device exists which can in fact switch instantaneously.

(ii) When generalized and applied to flexible structures, the discontinuous class of controls will likely excite poorly modeled higher frequency modes and

(iii) The predicted (model-derived) switch times and the response of the actual system are usually very sensitive to modeling errors.

The rigid body minimum-time, bang-bang control can be generalized to fomally control flexible body maneuvers, but *we have found bang-bang control of flexible body dynamics usually lacks robustness with respect to modeling errors.* An approach is desired which accommodates the inherent compromise between minimum-time, minimum-vibration, and maximum-robustness.

# Globally Stable Tracking Law

Assume that an open loop study has led to a control $u_{ref}(t)$. We seek the conditions satisfied by a perturbation feedback $\delta u(t)$ in $u(t) = u_{ref}(t) + \delta u(t)$ which guarantees global Liapunov stability with respect to the open loop path $\{\theta_r(t), y_r(t)\}$. Error energy fct.:

$$2U = a_0 I_{hub}\,\delta\theta^2 + \sum_{i=1}^{4} a_i \Big[ \int_{l_o}^{l_i} \rho_i \Big(\frac{\partial\,\delta y_i}{\partial t} + x_i\,\delta\theta\Big)^2 dx_i + \int_{l_o}^{l_i} EI_i \Big(\frac{\partial^2 \delta y_i}{\partial x_i^2}\Big)^2 dx_i + m_i \Big(l_i\,\delta\theta + \frac{\partial\,\delta y_i}{\partial t}\Big|_{l_i}\Big)^2 \Big] + a_5 (\delta\theta)^2 \quad (12)$$

Departure motion notation: $\delta(\,) = (\,) - (\,)_r$, or

$\delta(\,)$ = (exact quantity with control $u(t)$) - (exact quantity with control $u_{ref}(t)$)

Differentiation of Eq. (12), substitution of the eq. of motion, and some calculus leads to

$$\dot{U} = \frac{dU}{dt} = \delta\dot{\theta}\,\Big[ a_0\,\delta u + a_5\,\delta\theta + \sum_{i=1}^{4} (a_i - a_0)\,\delta(l_o S_o - M_o)_i \Big] \equiv \delta(\textit{weighted } \underline{\textit{power}}) \quad (13)$$

Since we require that $\dot{U} \le 0$, we set the [ ] term to $-a_6\,\delta\dot{\theta}$ and this leads to $\dot{U} = -a_6\,\dot{\theta}^2$ and the globally stabilizing control law:

$$u(t) = u_{ref}(t) - \Big[ g_1\,\delta\theta(t) + g_2\,\delta\dot{\theta}(t) + \sum_{i=3}^{6} g_i\,\delta(l_o S_o(t) + M_o(t))_i \Big]. \quad (14)$$

*This elegant tracking law provides global Liapunov path stability;* it is exactly analogous to the previous result except all displacements are measured from instantaneous point on the reference trajectory. The tracking law is invariant with respect to the usual beam approximations. Potential trouble in toyland! *Minimum time implies a sense of urgency*! The $(\,)_r$ quantities often cannot be computed in near real time.

## Stable Tracking With Respect to an Approximate Trajectory

In recent developments, we have shown that the globally stable law

$$u(t) = u_{ref}(t) - \left[ g_1\, \delta\theta(t) + g_2\, \delta\dot{\theta}(t) + \sum_{i=3}^{6} g_i\, \delta(l_o\, S_o(t) + M_o(t))_i \right], \tag{14}$$

can be replaced by the control

$$u(t) = u_{ref}(t) - \left[ g_1\, \Delta\theta(t) + g_2\, \Delta\dot{\theta}(t) + \sum_{i=3}^{6} g_i\, \Delta(l_o\, S_o(t) + M_o(t))_i \right], \tag{15}$$

where we use the departure motion notation: $\delta(\,) = (\,) - (\,)_r$, $\Delta(\,) = (\,) - (\,)_{ref}$ , or

$\delta(\,)$ = (*exact* quantity with control $u(t)$) - (*exact* quantity with control $u_{ref}(t)$)

$\Delta(\,)$ = (*exact* quantity with control $u(t)$) - (*approx.* quantity with control $u_{ref}(t)$)

We have proven that a sufficient condition for global stability of the closed loop solution of the system, using the control law of Eq. (15), is that the following inequality holds:

$$\dot{U} = -a_0(\dot{\theta}-\dot{\theta}_r)\,\{g_2(\dot{\theta}-\dot{\theta}_r) + [g_1\,\Delta\theta + g_2\,\Delta\dot{\theta} + \sum_i g_i\,\Delta(l_o\, S_o - M_o)_i\,]\} \le 0 \tag{16}$$

A sufficient condition characterizing the region where Eq. (16) holds is the dominance of the first term in the { } term; this gives the inequality

$$|\dot{\theta}-\dot{\theta}_r| > \frac{1}{g_2}|g_1\,\Delta\theta + g_2\,\Delta\dot{\theta} + \sum_i g_i\,\Delta(l_o\, S_o - M_o)_i| \equiv \mu \tag{18}$$

For the case of identical beams and anti-symmetric in unison deformations, we have carefully validated this result both numerically and experimentally in our laboratory.

# CONCLUDING REMARKS

- A method has been developed for designing stable feedback laws for nonlinear distributed parameter systems: Applies *directly* to hybrid ODE/PDE systems:
  - Not necessary to discretize the structure to establish the control law & stability
    - Key Point: discretization & spillover do not corrupt stability arguments
  - The form of the control law and the stability proof can be established using power equations.
    - Analysis directed to formation of kinetic & potential energies, work rates; & velocity level kinematics. Key headache: uncontrollable subspaces?
    - Must be careful to include all working forces & moments
- We used the method to establish a near-minimum-time, globally stable feedback law for a nine body hub/appendage structure.
  - Analytical results ... Numerical Results ... Experimental Results
    - Established a methodology for selecting torque-shaped reference trajectory to comprimise between minimum time and vib. suppression
    - Easy to implement & it works well
- Research Issues:

  *Stability* vs *Optimality* vs *Robustness*
  *Controllability* and *Measures of Controllability* for nonlinear DPS

ATTACHMENT 2

# A Nonrecursive "Order N" Preconditioned Conjugate Gradient/lRange Formulation of MDOF Dynamics

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# A NONRECURSIVE "ORDER N" PRECONDITIONED CONJUGATE GRADIENT / RANGE SPACE FORMULATION OF MDOF DYNAMICS

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## ABSTRACT

While excellent progress has been made in deriving algorithms that are efficient for certain combinations of system topologies and concurrent multiprocessing hardware, several issues must be resolved to incorporate transient simulation in the control design process for large space structures. Specifically, strategies must be developed that are applicable to systems with numerous degrees of freedom. In addition, the algorithms must have a *growth potential* in that they must also be amenable to implementation on forthcoming parallel system architectures. For mechanical system simulation, this fact implies that

(ii) Algorithms are required that induce parallelism on a fine scale, suitable for the emerging class of highly parallel processors.

(iii) Transient simulation methods must be automatically load balancing for a wider collection of system topologies and hardware configurations.

This paper addresses these problems by employing a combination range space / preconditioned conjugate gradient formulation of multi-degree-of-freedom dynamics. The method described herein has several advantages. In a sequential computing environment, the method has the features that:

(i) By employing regular ordering of the system connectivity graph, an extremely efficient preconditioner can be derived from the "range space metric", as opposed to the system coefficient matrix.

(ii) Because of the effectiveness of the preconditioner , preliminary studies indicate that the method can achieve performance rates that depend linearly upon the number of substructures, hence the title "Order N".

(iii) The method is non-assembling, i.e., it does not require the assembly of system mass or stiffness matrices, and is consequently amenable to implementation on workstations.

Furthermore, the approach is promising as a potential parallel processing algorithm in that

(iv) The method exhibits a fine parallel granularity suitable for a wide collection of combinations of physical system topologies / computer architectures.

(v) The method is easily load balanced among processors, and does not rely upon system topology to induce parallelism.

## (1.0) INTRODUCTION

There is no doubt that an effective design process for the space station absolutely requires that high fidelity simulations of the transient response to control inputs be rapidly attainable. Much research has been carried out over the past few years that concentrates on improving the performance of methods for simulating the dynamics of nonlinear, multibody systems [Gluck],[Haug],[Singh]. The research has primarily been devoted to

(i) the derivation of more efficient formulations of multibody dynamics, and to

(ii) the derivation of parallel processing algorithms.

Perhaps the most significant research addressing these two areas has been the introduction of the recursive, Order N algorithms in [Hollerbach],[Featherstone], and their subsequent refinements in [Bae], [Singh] for systems of rigid bodies. As noted in [Singh], these methods have the feature that the computational cost of the solution procedure is linear in the number of degrees of freedom N of the system, while conventional Lagrangian formulations are of cubic order. The conclusion that the Lagrangian methods are of cubic order derives from the fact that a system generalized mass/inertia matrix of dimension N X N must be factored at each time step. Just as importantly, the computational structure of the recursive Order N algorithms is amenable to parallel computation for some system topologies. If the system to be modelled has many independent branches in its system connectivity graph, the computational work required by the algorithm can be distributed among processors by assigning branches to independent processors. As an example, figure (1.1) illustrates the connectivity graph for an all terrain vehicle modelled in [Bae]. The processor computational load distribution employed in the paper is likewise illustrated. Because of the system connectivity and specific hardware architecture, excellent performance improvements and processor utilization are achieved in [Bae].

Due to these successes for rigid body simulations, it is well-known that many research institutions are presently investigating adaptations of the original recursive method to model systems comprised of flexible bodies. No doubt, the result will be highly efficient algorithms that perform well. Still, three key goals must be resolved before a general parallel processing algorithm can be obtained.

(i) Algorithms are required that induce parallelism on a finer scale, suitable for the emerging class of highly parallel processors.

(ii) Concurrent transient simulation methods must be automatically load balancing for a wider collection of combinations of mechanical systems and concurrent multiprocessing hardware.

(iii) The transient simulation method should also be amenable to vector processing implementation on each independent concurrent multiprocessor.

Based upon preliminary investigation, these goals should be very challenging if the algorithm is based upon an recursive Order N formulation.

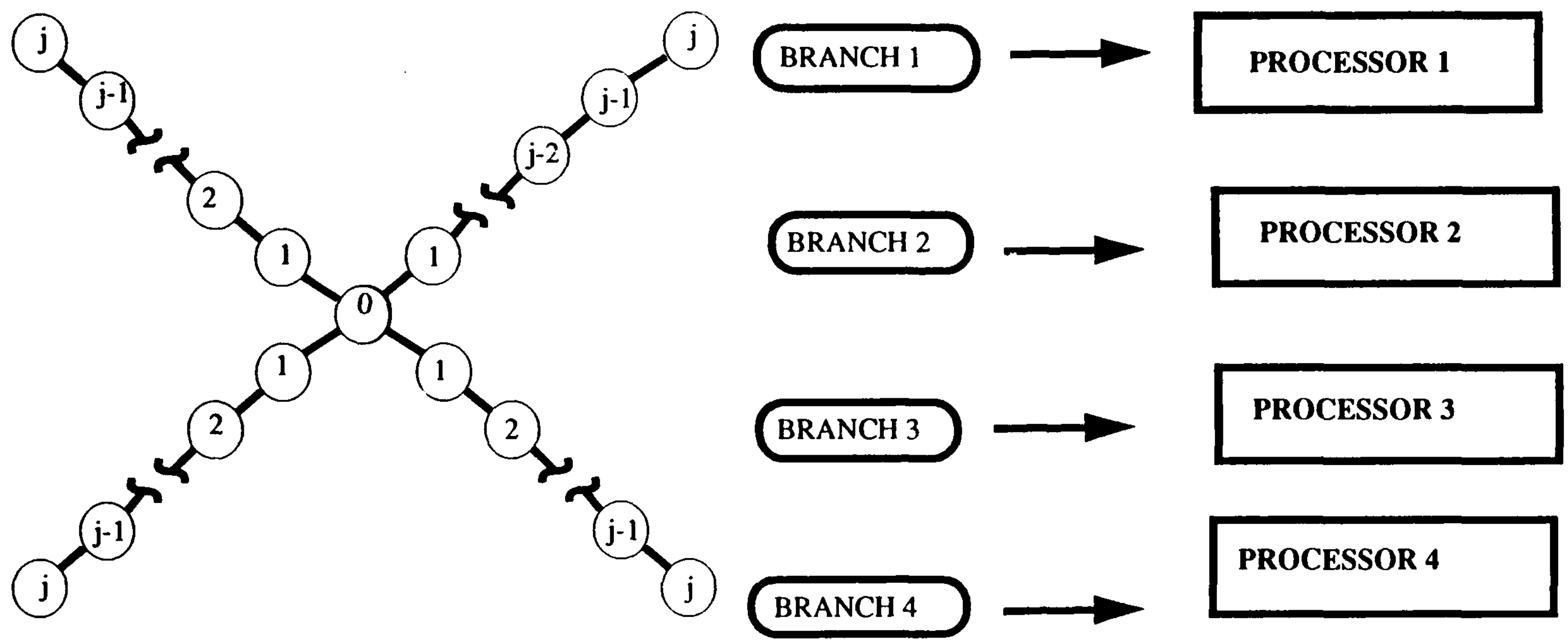



FIGURE (1.1) - PROCESSOR ASSIGNMENT

An innovative strategy based upon these goals is derived in this paper. In part, its foundation can be traced to element-by-element methods already in use in finite element solution procedures [Hughes]. As regards sequential computing environments:

(i) The combination range space formulation / PCG solution is an extremely efficient sequential algorithm for a class of problems described in the paper. The efficiency is primarily due to the selection of a Block Jacobi preconditioner that is rapidly convergent.

(i) The method is non-assembling, i.e. , it does not require a large amount of in-core storage, and consequently is also attractive as a candidate for implementation on workstations.

(iii) Preliminary studies indicate that due to the rapid convergence achieved by using the selected preconditioner, the method can achieve performance rates that depend linearly upon the number of substructures.

Moreover, the method should be readily implemented on parallel processors:

(iii) A vast literature exists on the amenability of the PCG solution procedure to both concurrent and vector processing.

(iv) The method is relatively easily load balanced among processors, and does not rely upon system topology to induce parallelism.

This paper focuses on the fundamental dynamical formulation using a combination range space / PCG solution, and its performance on sequential computing machines. Although the potential application of the method on parallel architectures is outlined, the details of a concurrent implementation are presented in a forthcoming paper.

## (2.1) RANGE SPACE / PRECONDITIONED CG EQUATIONS

The range space formulation of dynamics has been derived in the aerospace and mechanism dynamics research literature in [Placek],[Agrawal],[Kurdila]. Its theoretical foundation can be traced to the range space formulation of constrained quadratic optimization [Gill]. Still, despite the fact that it is often less computationally expensive than the nullspace methods, the nullspace method seems to have received more attention in the literature [Singh, Wehage, Kim, Huston, Kurdila...]. If the dynamics of a nonlinear, multibody system are governed by the collection of differential-algebraic equations

$$[M(q)\ddot{q} = f(q, \dot{q}, t) + [C(q)]^T \lambda$$

subject to constraints in linear, non-holonomic form

$$[C[q]]\ \dot{q} = 0$$

the range space solution of these equations are given by explicitly solving for the multipliers

$$\lambda = -([C(q)][M(q)]^{-1}[C(q)]^T)^{-1}\left\{[C(q)][M(q)]^{-1}f(q, \dot{q}, t) - e(q, \dot{q}, t)\right\}$$

and substituting to achieve a govering system of ordinary differential equations.

$$\ddot{q} = [M(q)]^{-1}\{f(q, \dot{q}, t) - [C(q)]^T$$
$$\left\{([C(q)][M(q)]^{-1}[C(q)]^T)^{-1}\left\{[C(q)][M(q)]^{-1}f(q, \dot{q}, t) - e(q, \dot{q}, t)\right\}\right\}$$

In the above equations, the constraints have been differentiated twice to yield

$$[C(q)]\ddot{q} = -\frac{d}{dt}([C(q)])\dot{q} = e(q, \dot{q}, t)$$

Any standard explict-predictor / implicit-corrector, or Runge-Kutta integration scheme can be applied to these equations provided that the condition number

$$\kappa(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$$

of the constraint metric

$$[C(q)][M(q)]^{-1}[C(q)]^T$$

does not become too large. The restriction that the condition number above remains small precludes the possibility of redundant constraints (for example, as associated with singularities arising from closed loops) and remains an underlying assumption throughout the rest of the paper.

One advantage of the range space equations for systems having many independent structures to be assembled is that the system coefficient matrix is block diagonal and, consequently, the factorization and back-substitution required to form the product of the inverse of the mass matrix and a given vector is relatively inexpensive to calculate. It requires that one calculate the factorization of the individual substructure mass matrices alone. In fact, one need not even assemble the system mass matrix, and the factorizations can occur in parallel. Unfortunately, if one subdivides the overall system into finer collections of substructures (to facilitate the factorization of the system coefficient matrix), numerous constraints are introduced into the model. As a consequence, one has the tradeoff shown in figure (2.1.1).

The approach taken in this paper is to finely subdivide the system to be modelled, and thus accrue the benefits of having a system coefficient matrix with smaller block diagonals, but also employ a solution procedure that ameliorates the cost associated with the increasing dimensionality of the constraint metric. Specifically, the calculation of the Lagrange multipliers in

$$\lambda=-([C(q)][M(q)]^{-1}[C(q)]^{T})^{-1}\{[C(q)][M(q)]^{-1}f(q,\dot{q},t)-e(q,\dot{q},t)\}$$

is carried out using the preconditioned conjugate gradient procedure.

## (2.2) THE PRECONDITIONED CONJUGATE GRADIENT SOLUTION

The preconditioned conjugate gradient procedure is an "accelerated" variant of the classical conjugate gradient procedure. If it is required to solve the linear system of equations

$$Ax = b$$

the procedure can be summarized from [Golub]

$$
\begin{array}{l}
x_0 = 0 \\
r_0 = b \\
\textbf{For } k = 1, \ldots n \\
\quad \textbf{If } r_{k-1} = 0 \\
\qquad \textbf{then} \\
\qquad\quad x = x_{k-1} \\
\qquad \textbf{else} \\
\qquad\quad \textbf{Solve } Qz_{k-1} = r_{k-1} \\
\qquad\quad \beta_k = z_{k-1}^T r_{k-1} / z_{k-2}^T r_{k-2} \\
\qquad\quad p_k = z_{k-1} + \beta_k p_{k-1} \\
\qquad\quad \alpha_k = z_{k-1}^T r_{k-1} / p_k^T A p_k \\
\qquad\quad x_k = x_{k-1} + \alpha_k p_k \\
\qquad\quad r_k = r_{k-1} - \alpha_k A p_k
\end{array}
$$

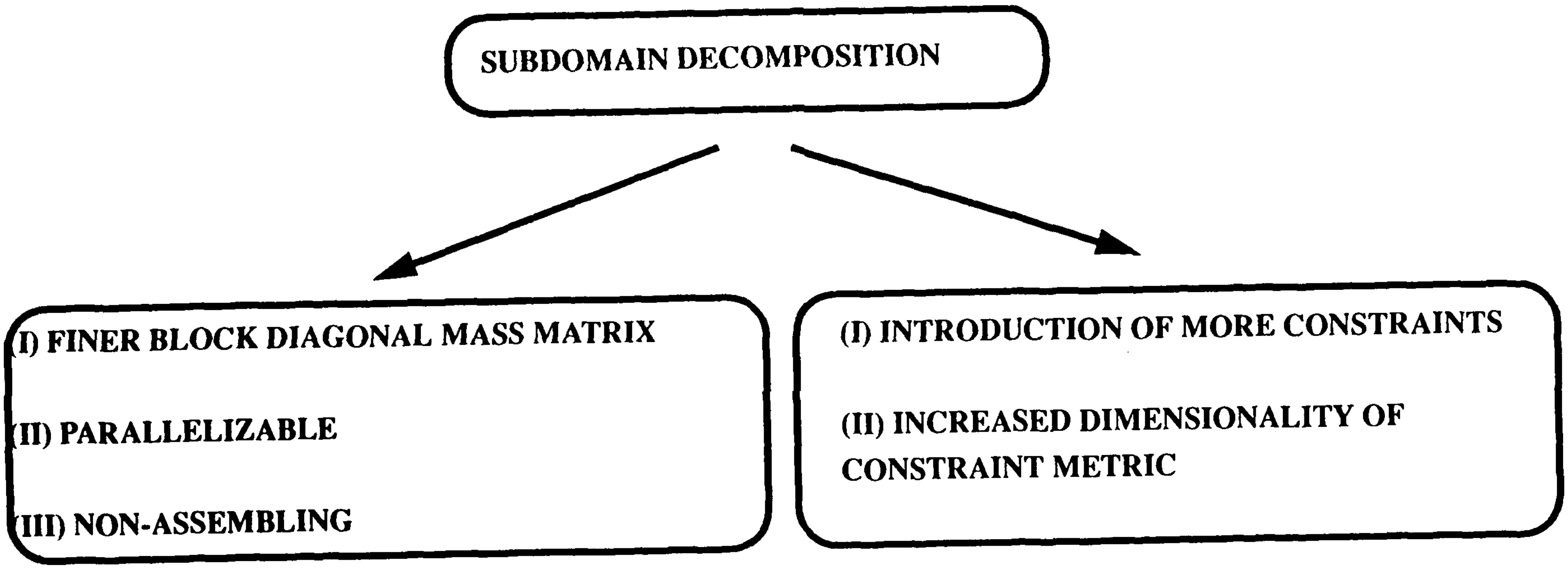



FIGURE (2.1.1) : COMPUTATATIONAL TRADEOFF DUE TO SUB DOMAIN DECOMPOSITION

Careful inspection of the algorithm shows that the most computationally expensive tasks in the procedure are the

(i) calculation of the product of the coefficient matrix $\boldsymbol{A}$ and a given residual vector,

(ii) and the solution of a linear system of equations requiring the factorization of the preconditioner $\boldsymbol{Q}$.

The rate of convergence of the preconditioned conjugate gradient algorithm is accelerated by employing a user-defined "preconditioning matrix." This matrix must have two properties to be an effective preconditioner:

(i) It must be relatively easy to factor.

(ii) It must be an approximate inverse to the constraint metric in a sense to be made precise below.

The reason for employing the preconditioned conjugate gradient solution method is that the convergence rate of the *conjugate gradient algorithm* (that is, with $\boldsymbol{Q} = I$) is governed by

$$\frac{\left\|\boldsymbol{x} - \boldsymbol{x}_k\right\|_{\boldsymbol{A}}}{\left\|\boldsymbol{x} - \boldsymbol{x}_0\right\|_{\boldsymbol{A}}} \leq \left[\frac{1 - \sqrt{\kappa(\boldsymbol{A})}}{1 + \sqrt{\kappa(\boldsymbol{A})}}\right]^{2k}$$

Thus, the rate of convergence of the algorithm improves as the condition number

$$\kappa(\boldsymbol{A})$$

decreases. The reference [Golub] has shown that the convergence of the preconditioned conjugate gradient method is governed by the same expression, but with $\boldsymbol{A}$ replaced with

$$\tilde{\boldsymbol{A}} = \boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{Q}^{-\frac{1}{2}}$$

Clearly, if the preconditioner is identical to the coefficient matrix, then the condition number of $\tilde{\boldsymbol{A}}$ is minimized. Hence, the preconditioner is sought such that its inverse approximates the inverse of the coefficient matrix. Many methods exist for the calculation of preconditioners [Golub]. It should be noted that while the motivation for the use of many of these preconditioners is mathematically sound, the final choice invariably involves some heuristic.

## (2.3) THE CHOICE OF THE PRECONDITIONER

The choice of the preconditioner employed in this paper is based upon the following assumptions regarding the structural/mechanical system to be modelled:

(i) The system closely resembles a series of chains of bodies

(ii) The number of interface degrees of freedom is small relative to the number of interior degrees of freedom for a substructure.

(iii) The system does not contain any closed chains.

To a large extent, these assumptions have been driven by the physical structure of the space station in its assembly complete configuration.

The preconditioner for the system constraint metric is based upon the topology of a chain of substructures as shown in figure (2.3.1). If

$$[C_i(q)] \in R^{d_i \times N}$$

denotes the constraint matrix connecting two bodies at the ith interface, the system constraint matrix has the form

$$[C(q)] = \begin{bmatrix} [C_1] \\ [C_2] \\ \vdots \\ \\ [C_k] \end{bmatrix} \in R^{D \times N}$$

The system constraint metric can then be written

$$\begin{bmatrix} [c_1][M]^{-1}[c_1]^T & \cdots & [c_1][M]^{-1}[c_k]^T \\ [c_2][M]^{-1}[c_1]^T & & \\ [c_3][M]^{-1}[c_1]^T & & \\ \vdots & & \\ & & \\ [c_k][M]^{-1}[c_1]^T & \cdots & [c_k][M]^{-1}[c_k]^T \end{bmatrix}$$

Based upon the structure of the constraint metric above, the preconditioner is selected to be the block diagonal matrix

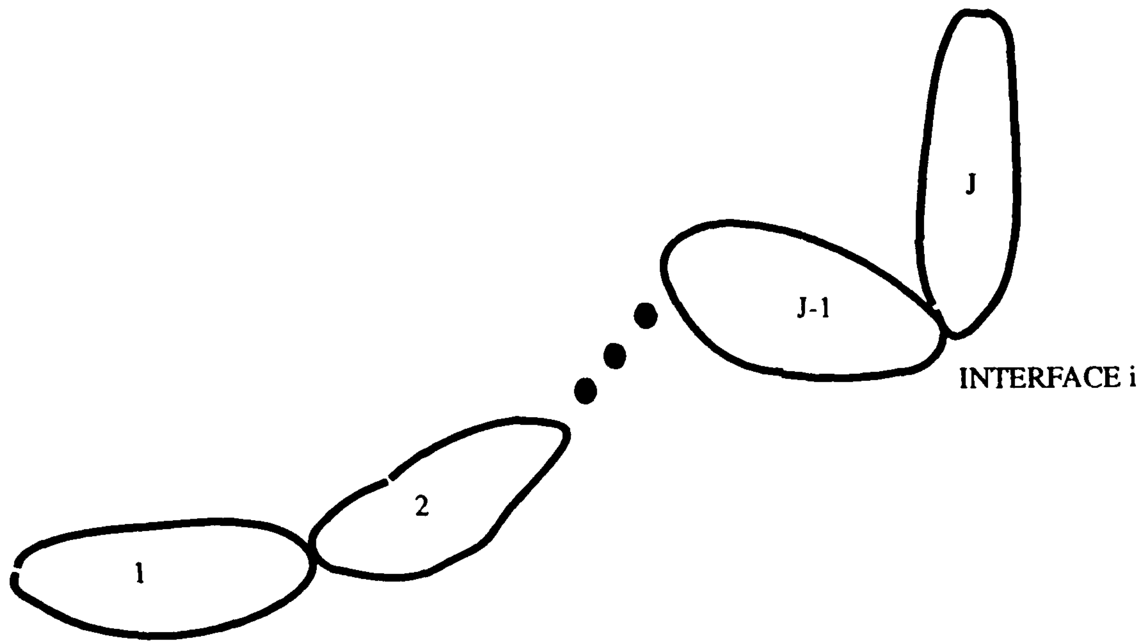



FIGURE (2.3.1) : MULTIBODY CHAIN

$$\begin{bmatrix} [c_1][M]^{-1}[c_1]^T & & & \\ & [c_2][M]^{-1}[c_2]^T & & \\ & & \ddots & \\ & & & [c_k][M]^{-1}[c_k]^T \end{bmatrix}$$

Although the off-diagonal blocks

$$[C_i(q)][M(q)]^{-1}[C_j(q)]^T = 0$$

(for i not equal to j) are not generally identically equal to zero, this choice of preconditioner is shown to be extremely efficient for the class of problems described in the next section. Furthermore, this preconditioner satisfies the two essential criteria of good preconditioners:

(i) It is block diagonal, with small diagonal blocks, and is relatively easy to factor.

(ii) It has an inverse that provides a good approximation to the inverse of the full system coefficient matrix.

This latter conclusion results from the well-known fact [Wittenburg] that the directed graph representing the connectivity of an open loop system can be regularly ordered. The regular ordering results in a system constraint metric that has a reduced bandwidth. That is, many of the off-diagonal blocks

$$[C_i(q)][M(q)]^{-1}[C_j(q)]^T = 0$$

are identically zero for i>>j. The choice of preconditioner shown above is often denoted the Block Jacobi preconditioner and is known to be highly effective for classes of systems of equations arising from elliptic partial differential equations [Reid].

## (3.1) SPACE MAST SIMULATIONS

To establish the efficiency and performance of the combined range space / PCG algorithm, several transient simulations have been carreid out. The first such simulation has been designed to answer two key questions regarding the feasibility of the approach for multibody structures:

(i) How efficient is the selected preconditioner and how does the preconditioner improve the effectiveness of the conjugate gradient iteration processes as a whole?

(ii) How efficiently can the range space / PCG solution for the transient response be carried out? In particular, what is the computational cost of solving the "constraint metric" factorization at each time step?

These two questions, that is, preconditioner efficiency and constraint metric factorization were judged to the crucial computational "bottlenecks" for the algorithm as a whole.

The two substructures selected for evaluation of the algorithm in studying its feasibility are shown in figures (3.1.1) and (3.2.2). The first structure is a 63 degree of freedom Z-truss substructure comprised of rod elements generated by MSC PAL. The second assembly is a 54 degree of freedom X-frame substructure comprised of three dimensional beam elements generated by the same program. Figures (3.1.3) and (3.1.4) illustrate that a remarkable convergence rate is achieved using the preconditioner derived from the regularly ordered graph of the constraint metric. In figure (3.1.3) the number of iterations required by the preconditioned conjugate gradient algorithm to converge to a tolerance of 1.e-14 of the true solution is plotted versus the number of degrees of freedom on the horizontal axis. Thus, the number of flexible degrees of freedom on the horizontal axis varies from 126 to 504 as substructures are contilevered end-to-end. As clearly illustrated in the diagram, the number of preconditioned conjugate gradient iterations remains constant, and equal to 3, independent of the number of total degrees of freedom. The other line plotted on the graph is the analytical upper limit to the number of iterations required in the conjugate gradient method. Completely analogous results are depicted in figure (3.1.4). In this case the number of total flexible degrees of freedom in the structure varies from 108 to 324, and the number of iterations of the preconditioned conjugate gradient algorithm required to achieve convergence to a tolerance of 1.e-14 is plotted on the vertical axis. Again, independent of the number of degrees of freedom, the number of iterations remains constant and equal to 4. Consequently, one can conclude that the algorithm for generating the preconditioner is indeed extremely efficient, for the test problems simulated.

Figures (3.1.5) and (3.1.6) depict the total time required per time step to solve the "inversion" of the system constraint metric. Figure (3.1.6) depicts the corresponding results for the Z-truss, while figure (3.1.6) depicts the results for the X-frame. In both cases it is clear that the simulation grows linearly as a function of the total number of flexible degrees of freedom. To the authors' knowledge, no such result for flexible bodies has been presented to date.

In later sections, while considering the amenability of the overall solution procedure to parallel processing, it is crucial to consider where computation time is spent during a typical time step. These results are shown in figure (3.1.7). This figure shows that two steps dominate the time spent solving the range space / PCG solution procedure:

(i) the time spent forming the product of the inverse of the system mass matrix and a given vector, and

(ii) the time spent applying the inverse of the preconditioner to a given vector.

As will be discussed in more detail in a later section on applications in parallel processing, these two steps are easily parallelized because of their block diagonal structure, and should yield excellent improvements in performance on concurrent multiprocessors.

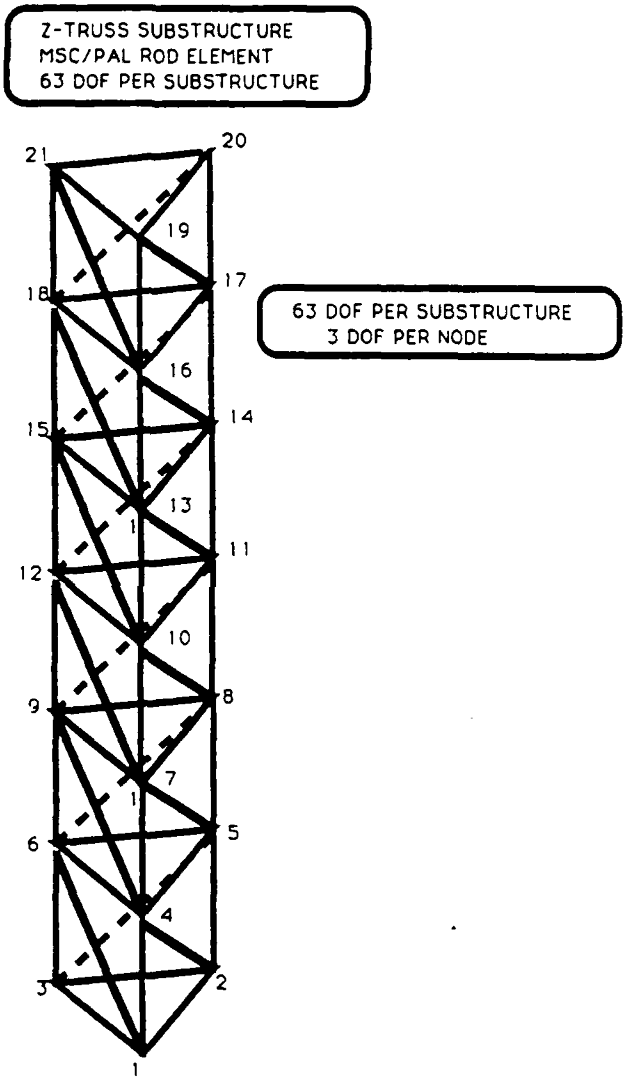



FIGURE (3.1.1) :63 DOF Z-TRUSS SUBSTRUCTURE

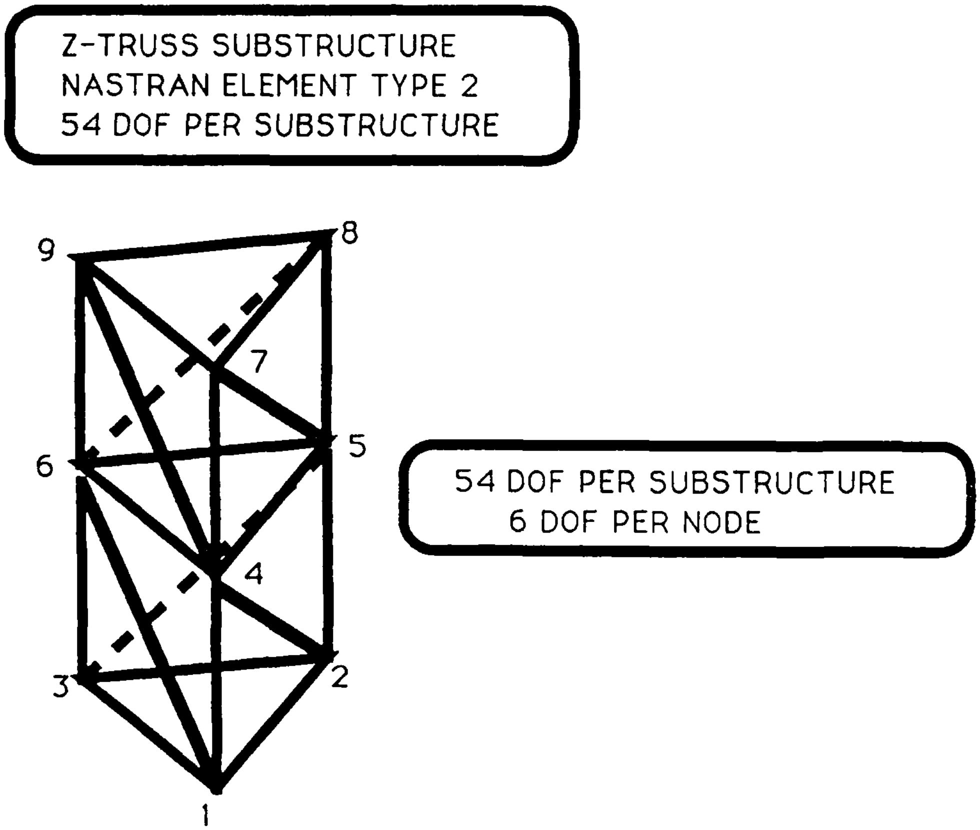
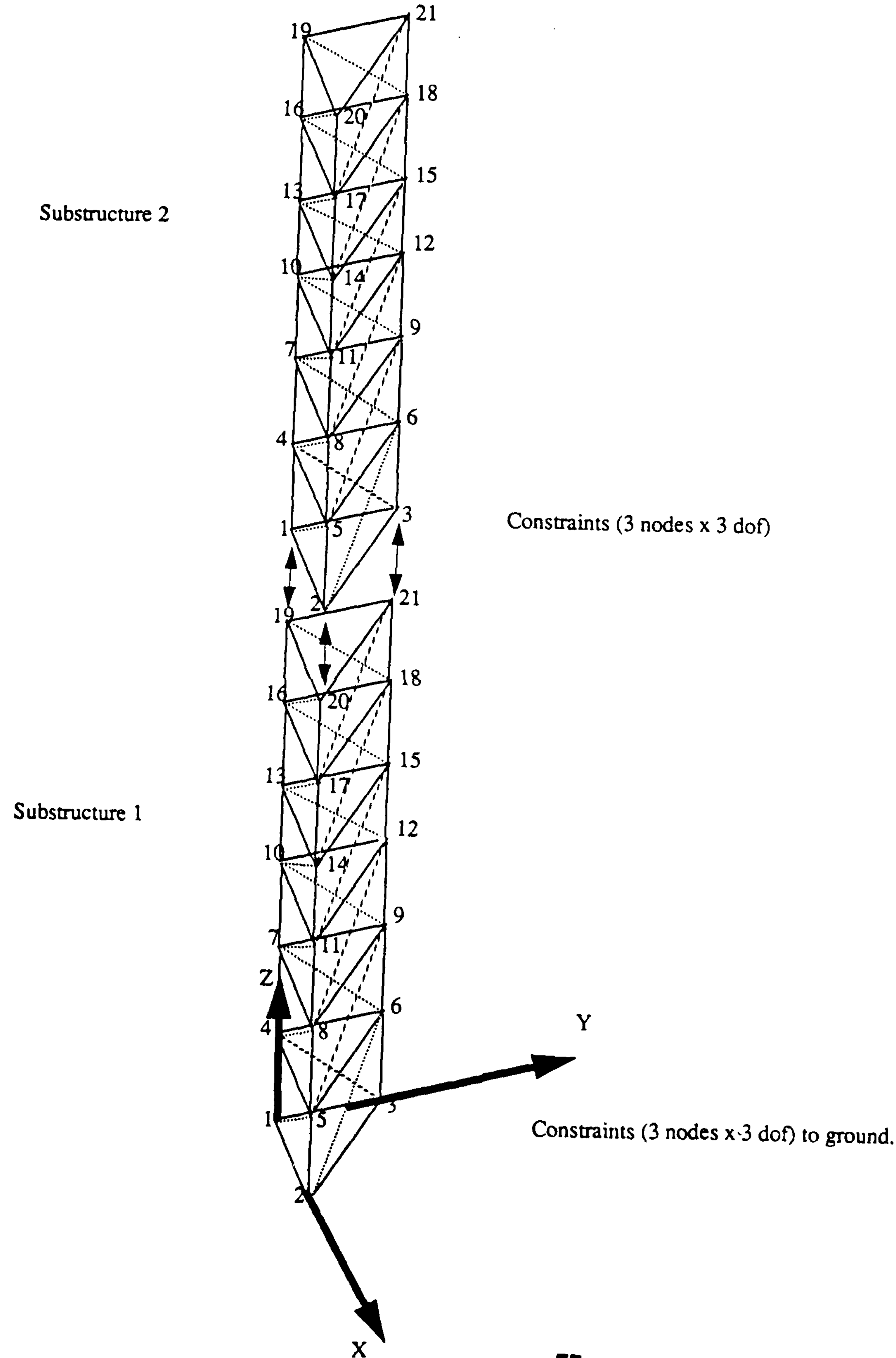



FIGURE (3.1.2) :54 DOF X-FRAME SUBSTRUCTURE


Substructure 2
Constraints (3 nodes x 3 dof)
Substructure 1
Constraints (3 nodes x 3 dof) to ground.
Z
Y
X

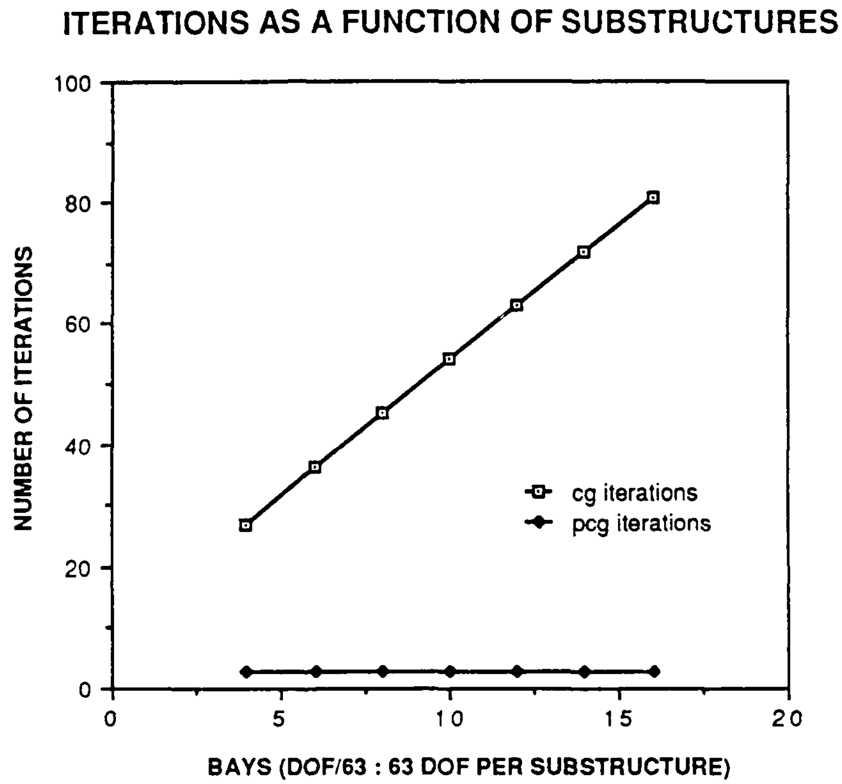



FIGURE (3.1.3) :ITERATIONS FOR Z-TRUSS

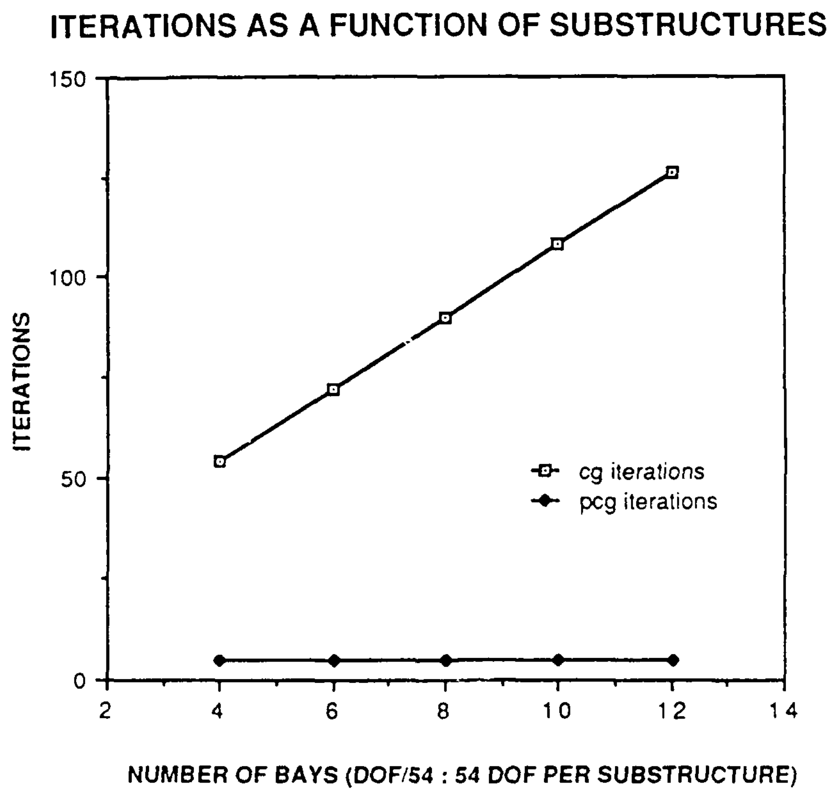



FIGURE (3.1.4) :ITERATIONS FOR X-TRUSS

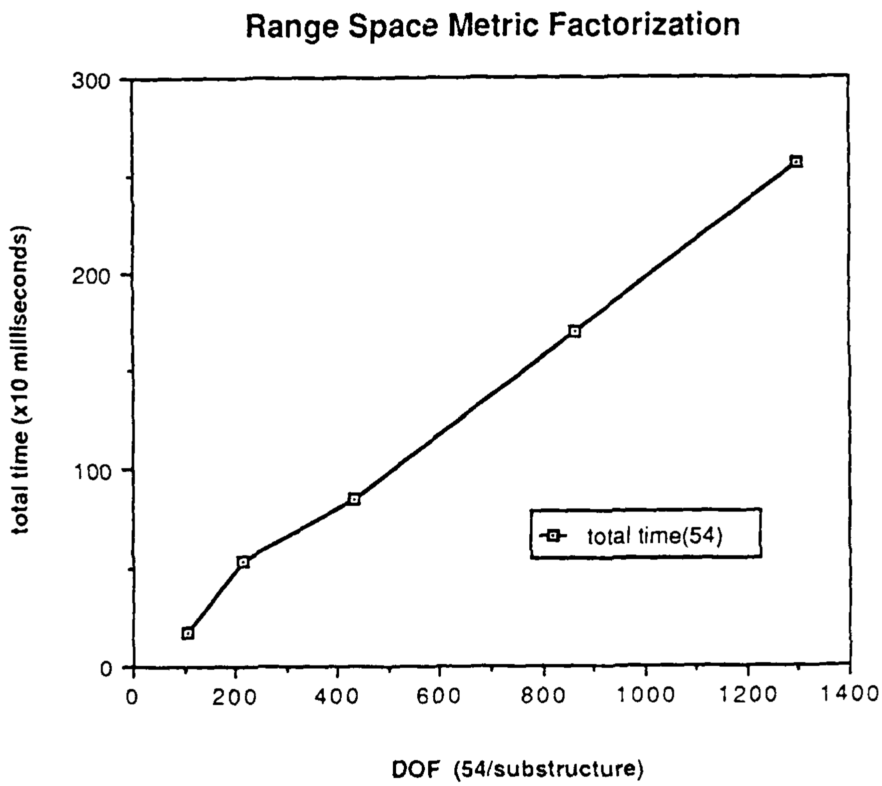



FIGURE (3.1.5) :SYSTEM CONSTRAINT METRIC SOLUTION FOR Z-TRUSS

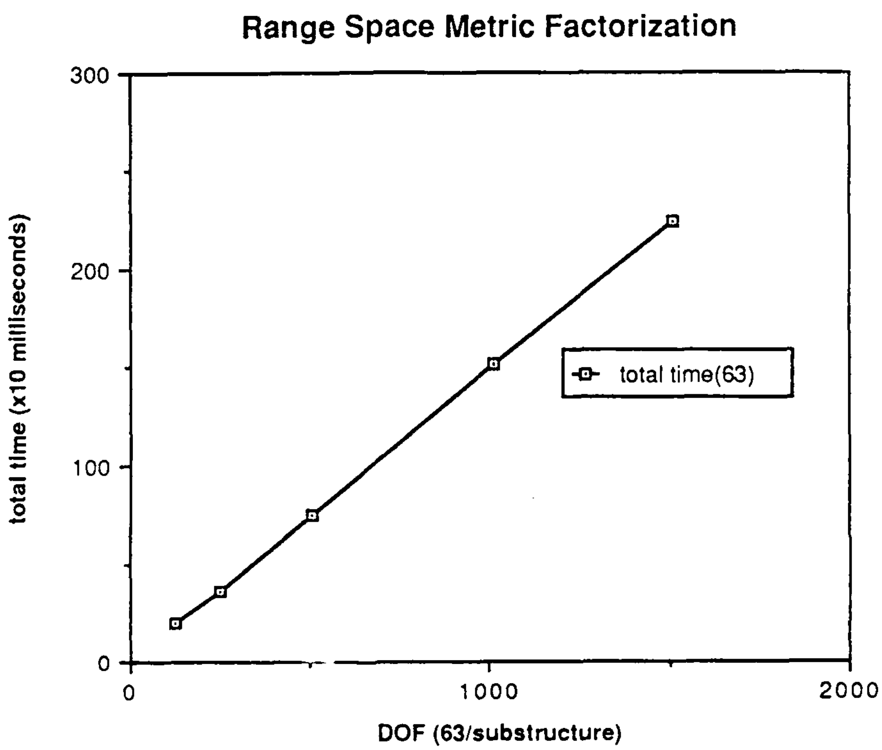



FIGURE (3.1.6) :SYSTEM CONSTRAINT METRIC SOLUTION FOR X-FRAME

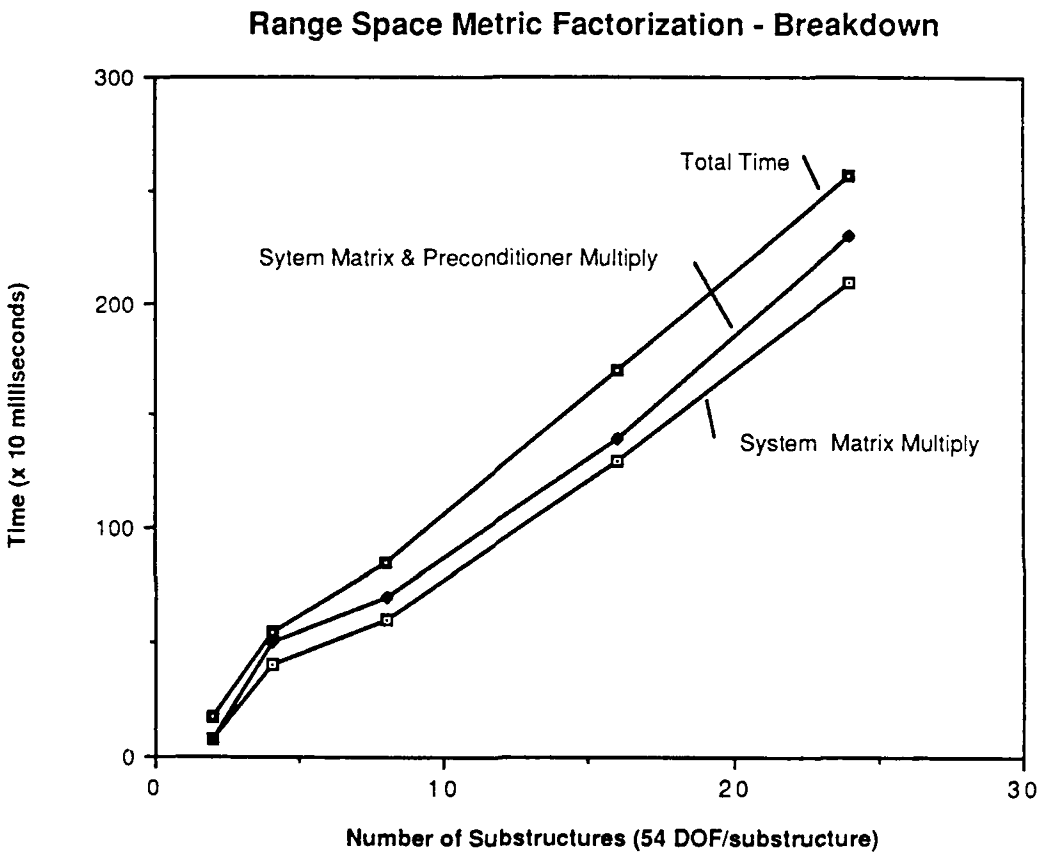



FIGURE (3.1.7) :BREAKDOWN OF TIMING RESULTS FOR 63 DOF Z-TRUSS

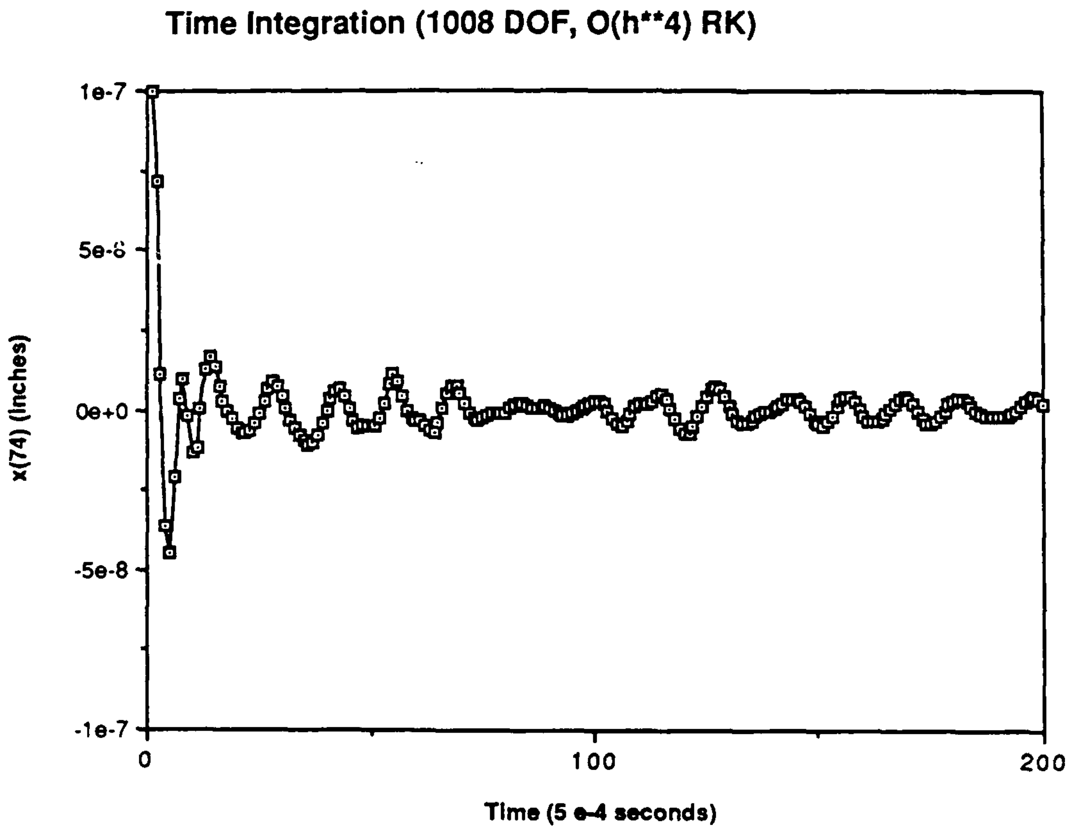



FIGURE (3.1.8) : TRANSIENT RESPONSE HISTORY OF DOF 74 (1008 DOF MODEL)

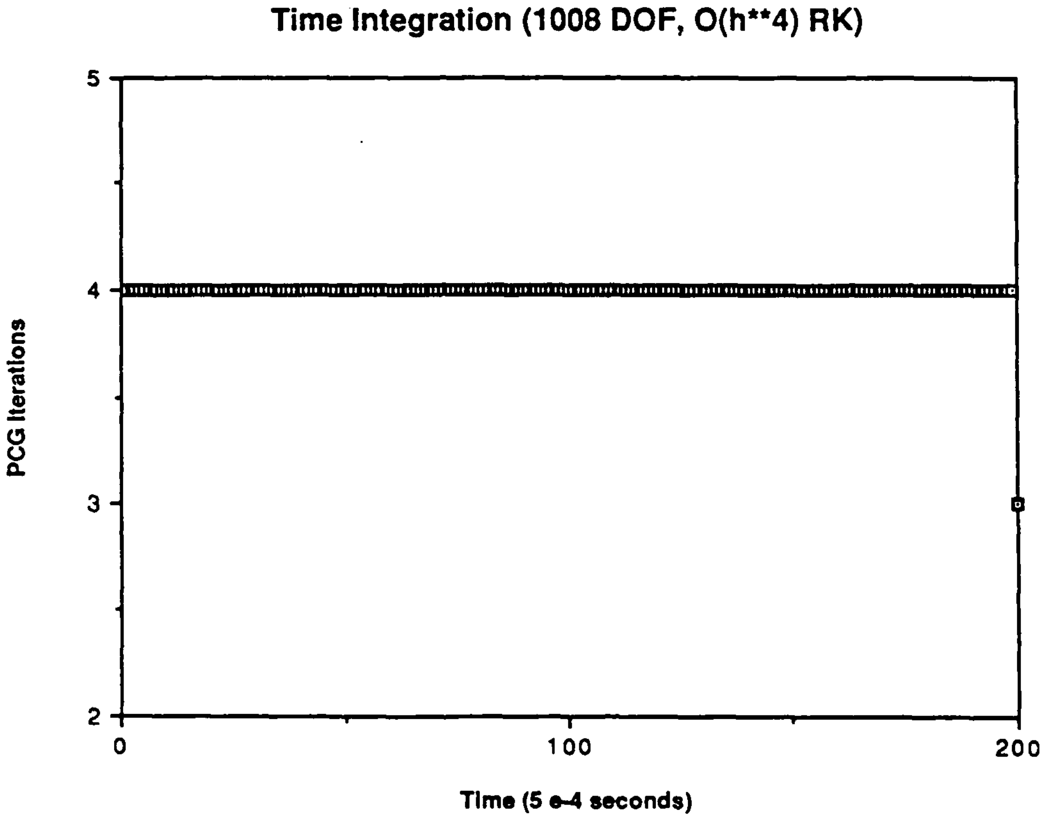



FIGURE (3.1.9) : PCG ITERATION TIME HISTORY / 63 DOF Z-TRUSS / 1008 DOF MODEL

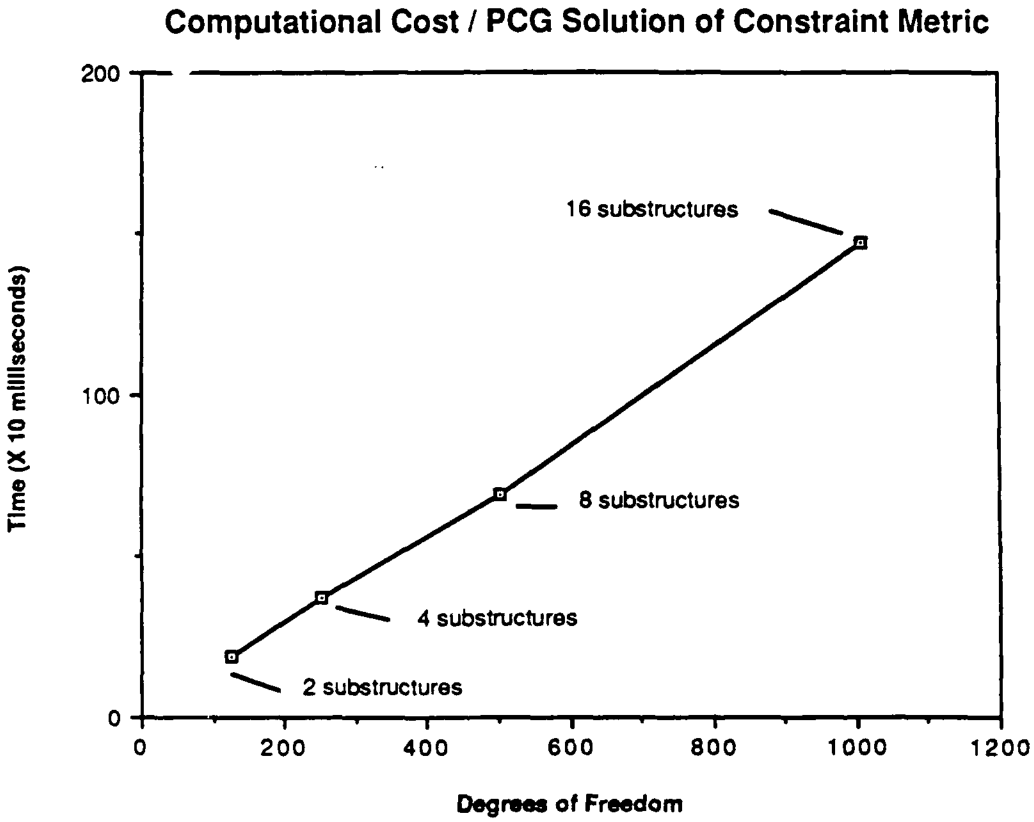



FIGURE (3.1.10) : COMPUTATIONAL COST / 63 DOF Z-TRUXX / 2,4,8,16 SUBSTRUCTURES

## (3.2) SPACE STATION SIMULATIONS

While the simulations of the truss structures in the last section answer important questions regarding the efficiency and performance of the PCG algorithm, it remains to be shown that the method is robust in problems that do not enjoy such structure in form. By analogy, one can liken the addition of identical substructures with identical interfaces to the solution of Poisson's equation on a rectangular grid: many methods exist that will have "remarkable" convergence rates for such highly structured problems. Unfortunately, the performance degrades rapidly as more general problems are considered.

In this section, the simulation of Space Station Freedom in the assembly complete configuration is considered. Qualitatively, the simulations described in this section differ from those in the last section in three important respects:

(i) the number of degrees of freedom per substructure varies,

(ii) the constituent substructures matrices are not identical in general (although some occur in symmetric pairs about the core body),

(iii) the number of degrees of freedom between substructure interfaces varies.

The model employed is a simplified version of the full finite element model for the assembly complete space station shown in figure (3.2.1). The space station is subdivided into 13 individual substructures having from 90 to 150 degrees of freedom each. The number of constraints per interface varies from 24 to 36. The first simulation considers only the central bodies 5 through 9. Blowups of these five bodies are shown in figures (3.2.2) through (3.2.6), and their position in the assembly complete station is shown in figure (3.2.7). A concise summary of the results of several simulations is given in figures (3.2.8) and (3.2.9). Figure (3.2.9) plots the total time per integration time step versus the number of degrees of freedom in the model. The model is assembled from left to right, so that data points are plotted for systems comprised of 2, 3, 4, and 5 substructures. The largest model considered is comprised of nearly 500 degrees of freedom. Because the number of degrees of freedom per body and number of constraints per interface varies with the addition of each set of bays, the computational cost plots are not exactly linear as in the space boom simulations. Still, a linear curve fit describes the timing data well for all the simulations considered. As in the case of the space boom simulations, a very large fraction of the overall simulation time is spent

(i) calculating a product of the inverse of the mass matrix and a given vector,

(ii) calculating the product of the inverse of the preconditioner and a given vector.

As noted earlier, both of these matrices are block diagonal and trivially parallelizable. Hence, the potential for implementing the method in a parallel environment is very promising. Figure (3.2.10) depicts the number of PCG iterations versus the integration time history. As in the previous example, the selected preconditioner is extremely effective, and requires 4 iterations for convergence, essentially independent of the number of degrees of freedom in the model.

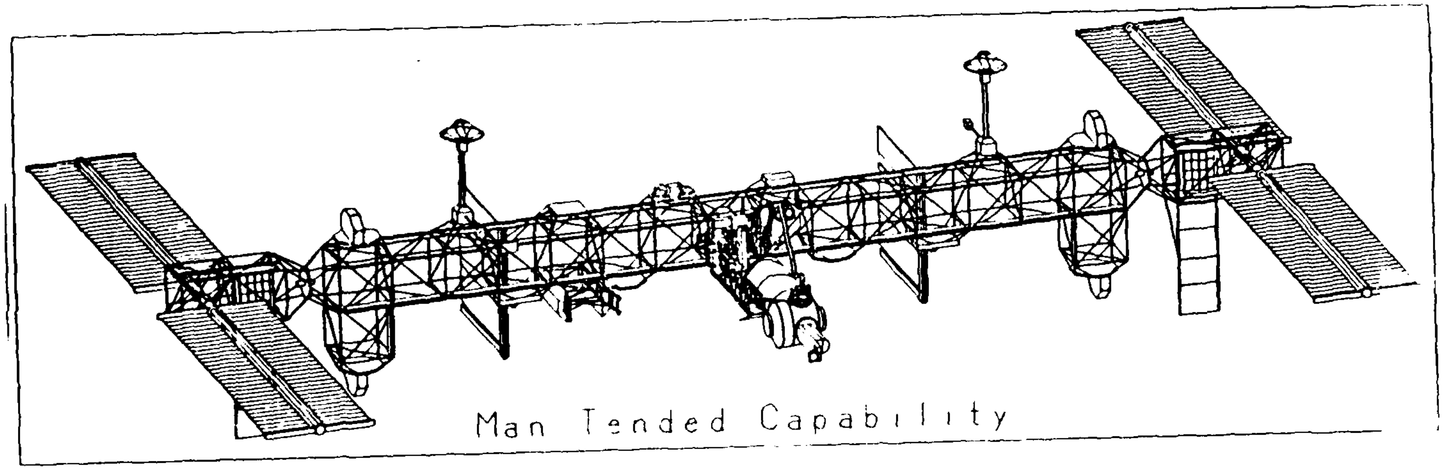



FIGURE (3.2.1) : ASSEMBLY COMPLETE SPACE STATION FREEDOM

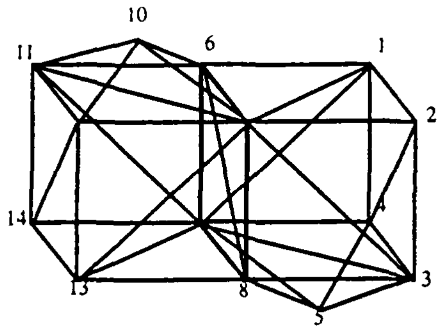



FIGURE (3.2.2) : SUBSTRUCTURE 5

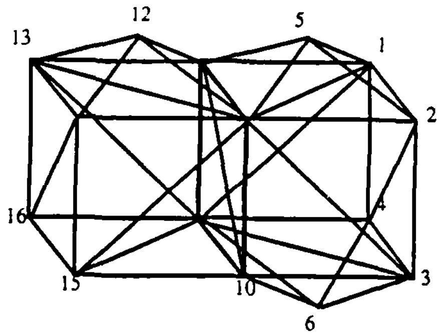



FIGURE (3.2.3) : SUBSTRUCTURE 6

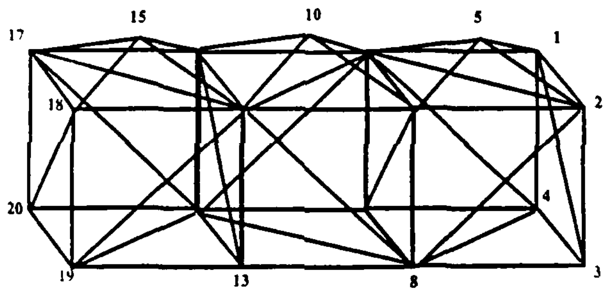



FIGURE (3.2.4) : SUBSTRUCTURE 7

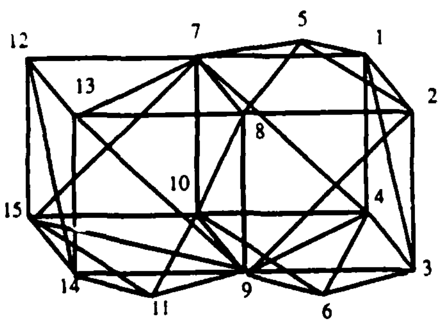



FIGURE (3.2.5) : SUBSTRUCTURE 8

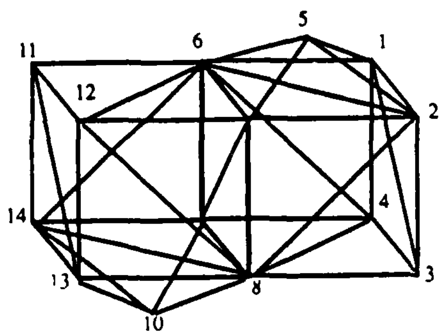



FIGURE (3.2.6) : SUBSTRUCTURE 9

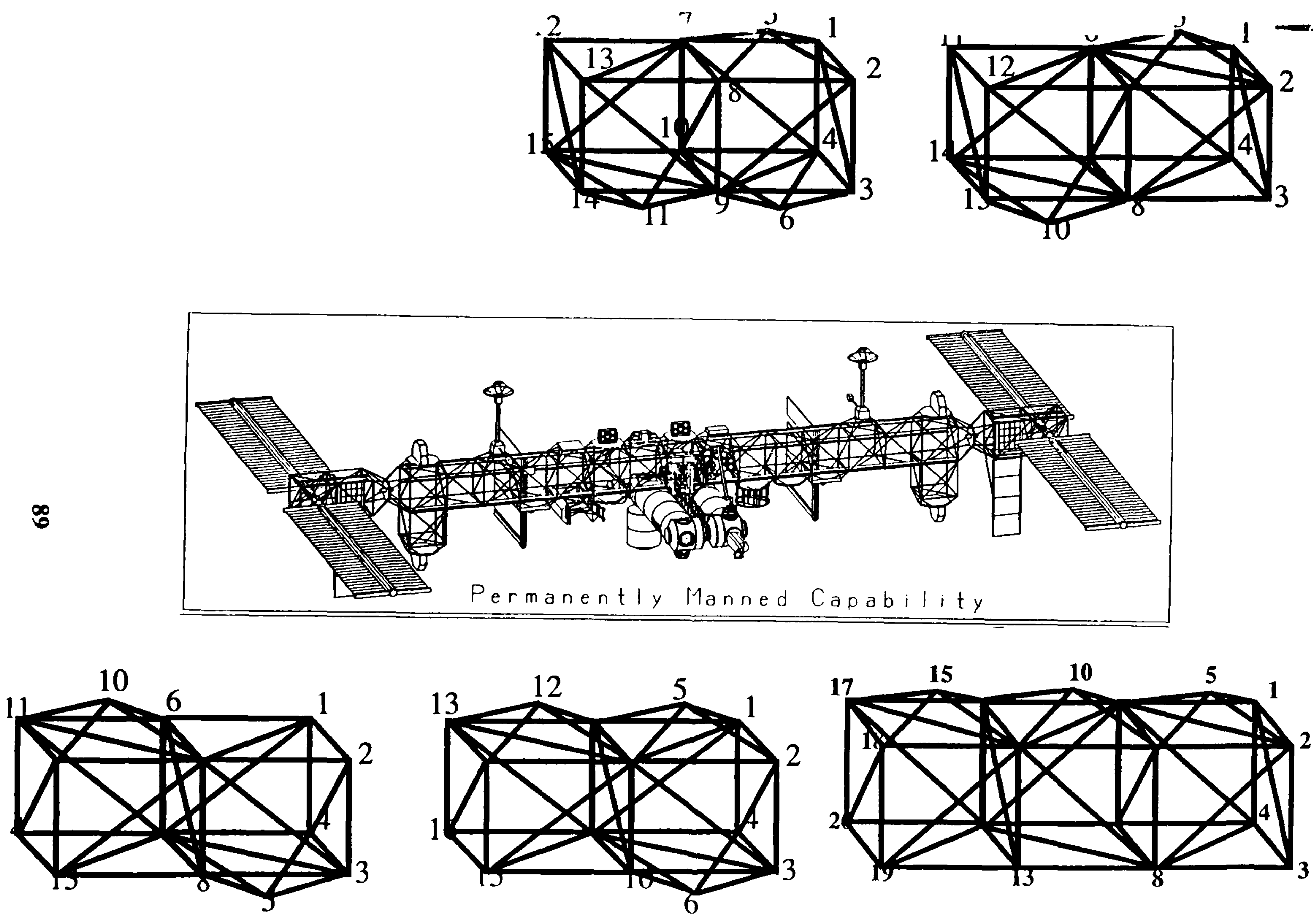



FIGURE (3.2.7) : SUBSTRUCTURE LOCATIONS IN ASSEMBLY COMPLETE CONFIGURATIONS

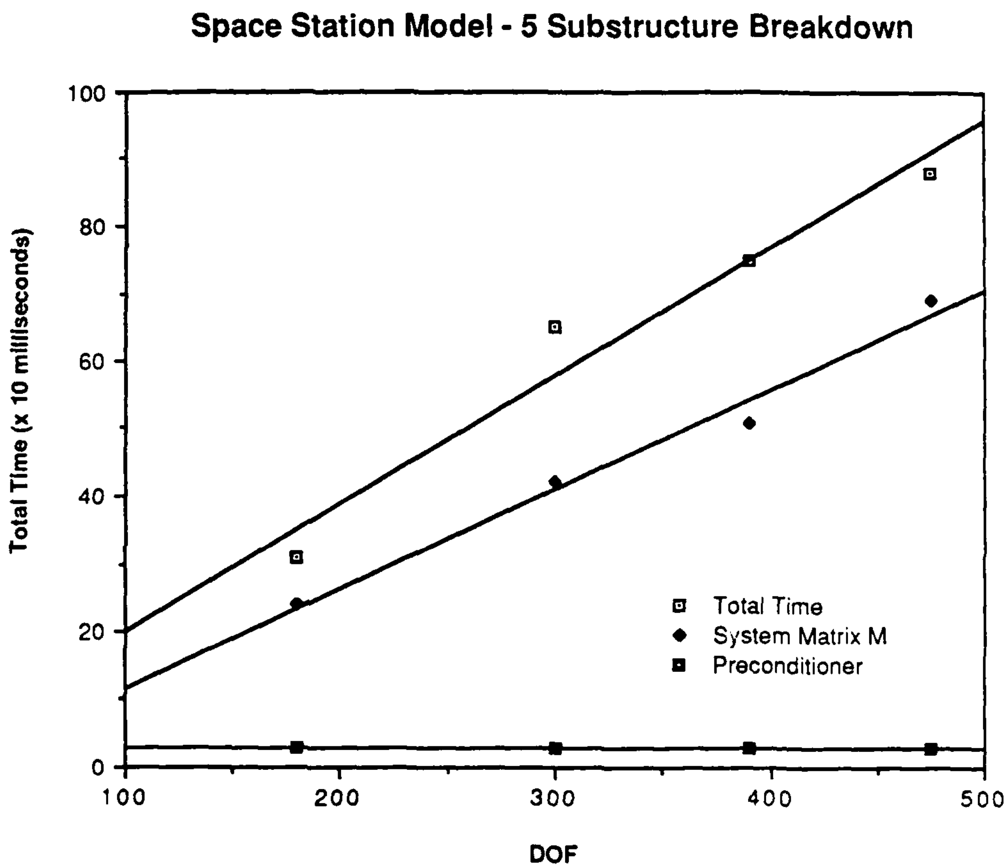



FIGURE (3.2.8) : SPACE STATION MODEL - 5 SUBSTRUCTURE TIMING BREAKDOWN

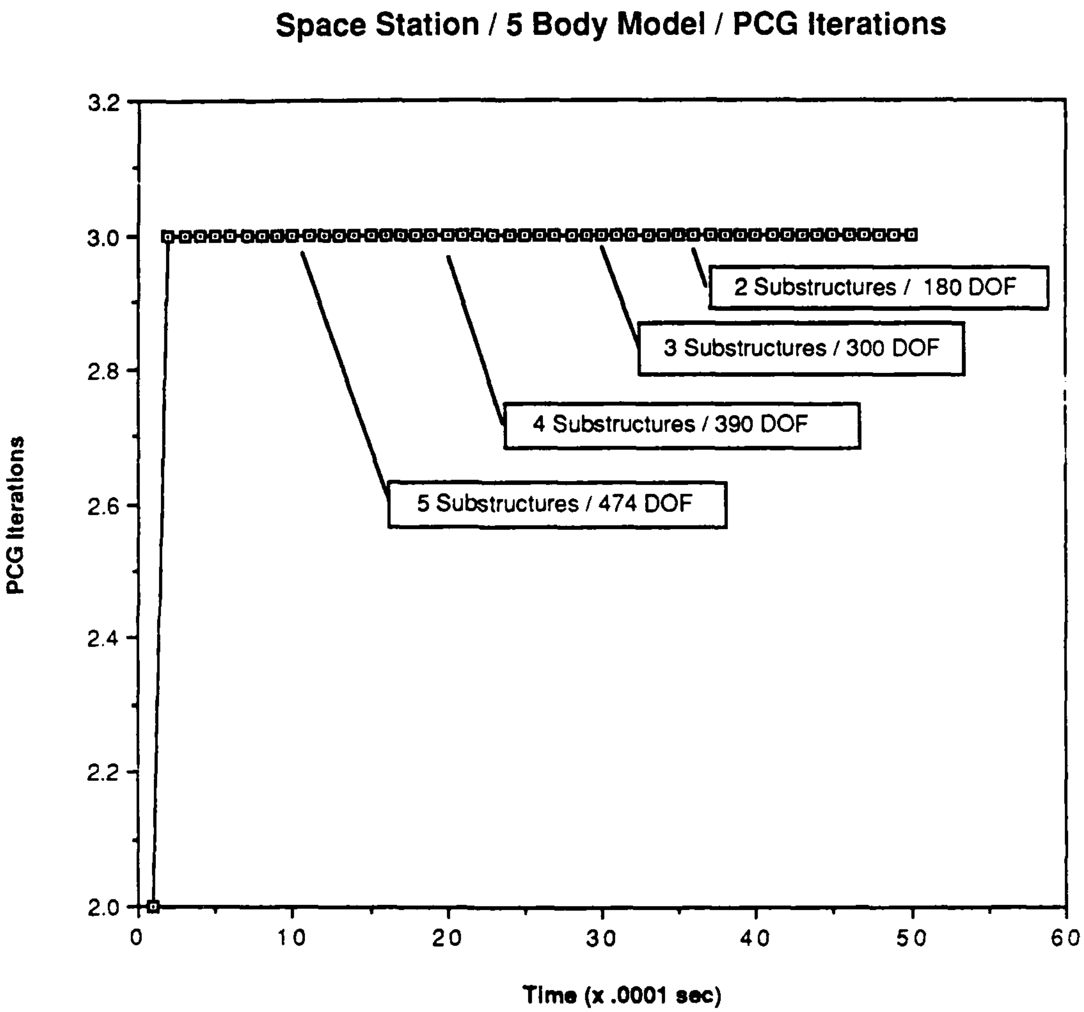



FIGURE (3.2.9) : PRECONDITIONED CONJUGATE GRADIENT ITERATION TIME HISTORY

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ATTACHMENT 3

# Time Domain Convergence Properties of Lyapunov Stable Penalty Methods

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# TIME DOMAIN CONVERGENCE PROPERTIES OF LYAPUNOV STABLE PENALTY METHODS

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## ABSTRACT

While earlier papers have studied the convergence properties of Lyapunov stable penalty methods (LSPM) as applied to spectral approximations, the current paper investigates the existence and convergence of penalty approximations when applied to transient analysis. This paper makes use of standard techniques in the analysis of linear, hyperbolic partial differential equations to show that a sequence of solutions generated by the Lyapunov stable penalty equations approaches the solution of the differential-algebraic equations (DAE's) governing the dynamics of multibody problems arising in linear vibrations. Specifically, the analysis relies upon

(1) the existence of Lyapunov functions for the class of problems considered,
(2) standard operator norm approximations of orthogonal projections onto the range of the transposed constraint matrix,
(3) the application of Gronwall's inequality to show that the sequence of approximate solutions remains bounded for compact intervals in time.

The analysis is quite general in that no assumption is made that the system be natural or completely integrable. The result of the analysis is the derivation of an explicit variational relationship between the norm of the constraint violation time history and the error between the solutions of the true and approximate penalty equations. For linear, undamped multi-degree-of-freedom equations this relationship takes the form

$$\lim_{\varepsilon \to 0} \{ \| K (X - X_\varepsilon) \| \} \le \alpha \| P X_\varepsilon \| \le \alpha_1 \| \Phi_\varepsilon \|$$

In the case of damped multi-degree-of-freedom equations, the variational relationship is

$$\lim_{\varepsilon \to 0} \{ \| C (\dot{X} - \dot{X}_\varepsilon) \| + \| K (X - X_\varepsilon) \| \} \le \alpha \| P X_\varepsilon \| + \mu \| P \dot{X}_\varepsilon \| \le \alpha_1 \| \Phi_\varepsilon \| + \mu_1 \| \dot{\Phi}_\varepsilon \|$$

## (1.0) INTRODUCTION

The difficulty that may arise in numerically integrating systems of differential-algebraic equations is well-documented [Gear],[Wehage]. Numerical procedures intentionally designed for as general classes of DAE's have been the topic of much research over the past two years by numerical analysts [Gear],[Burrage],[Alexander].

In computational mechanics, on the other hand, methods have arisen that utilize the fact that the particular system of differential-algebraic equations to be solved have been derived from Lagrangian or Hamiltonian formulations of dynamics. Examples of this type of approach are given in [Bayo 1,2,3], [Park], [Kurdila]. Essentially, all of these methods approximate the system of governing differential-algebraic equations by an altogether different dynamical system; one that has been obtained via penalty perturbation of the Lagrangian or Hamiltonian for the system.

While all of thes papers *present* considerable empirical evidence that the penalty methods are stable and convergent, little analysis has been conducted to establish this fact. Because the governing equations are neither linear, nor coercive in general, standard results as in [Oden 1] or [Oden 2] are not directly applicable. Simple, but effective error estimates for spectral approximations using the penalty method have been presented in [Kurdila]. Nonlinear stability and convergence criteria are considered in [Kurdila 1], [Kurdila 2], but rely upon the restrictions that the governing system be natural, and in some cases conservative. In the latter case, much of the arguments presented are based upon the underlying Hamiltonian structure of the systems considered.

The purpose of this paper is to investigate convergence criteria for linear multi-degree-of -freedom systems arising in linear vibrations. The analysis that follows is quite general in that it does not require that the system be conservative. No restriction on the form of the forcing term is enforced other than it is $L_2$-integrable on bounded intervals of time. The paper concludes by deriving variational statements that bound the error in approximation by the norm of the constraint violation obtained in the approximate solutions. These variational statements are of great practical importance: they imply that by monitoring the constraint violation one can be assured that the solution is accu rate. *One should note that this result is not true in general for nonlinear systems! For the nonlinear case, bifurcations as described in [Kurdila] can occur in which the constraint violation remains small, but the penalty solution diverges from the true solution.*

## (2.0) LINEAR, UNDAMPED EQUATIONS OF MOTION

### (2.1) Original Equations

As a starting point for the analysis, we consider the system of undamped, second order ordinary differential equations subject to holonomic constraints

$$\hat{M}x + \hat{K}x = \left[\frac{\partial\Phi}{\partial x}\right]^T \lambda + \hat{F}$$

where

$$\left[\frac{\partial\Phi}{\partial x}\right] x = 0$$
$$\left[\frac{\partial\Phi}{\partial x}\right] \dot{x} = 0$$
$$\left[\frac{\partial\Phi}{\partial x}\right] \ddot{x} = 0$$

and the dimensions of the constituent matrices and vectors are

$$x \in R^N \qquad \hat{F} \in R^N$$
$$\Phi \in R^D \qquad \hat{M} \in R^{N \times N}$$
$$\lambda \in R^D \qquad \hat{K} \in R^{N \times N}$$
$$\left[\frac{\partial\Phi}{\partial x}\right] \in R^{D \times N}$$

As is usually encountered in linear vibration equations, M is symmetric, positive definite, and may be considered to have been generated by consistent finite element formulation (as opposed to a lumped formulation). The stiffness matrix K is assumed to be symmetric positive semi-definite, and the constraint matrix is constant. To simplifiy the analysis, we introduce the following change of variables

$$K = \hat{M}^{-1/2} K \hat{M}^{-1/2}$$

$$X = \hat{M}^{1/2} x$$

$$\left[\frac{\partial \Phi}{\partial X}\right] = \left[\frac{\partial \Phi}{\partial x}\right]\left[\frac{\partial x}{\partial X}\right] = \left[\frac{\partial \Phi}{\partial x}\right]\hat{M}^{-1/2}$$

$$F = \hat{M}^{-1/2}\hat{F}$$

With these definitions, the governing equations take the simple form

$$\ddot{X} + KX = \left[\frac{\partial \Phi}{\partial X}\right]^T \lambda + F$$

while the constraints become

$$\left[\frac{\partial \Phi}{\partial X}\right] X = 0$$

$$\left[\frac{\partial \Phi}{\partial X}\right] \dot{X} = 0$$

$$\left[\frac{\partial \Phi}{\partial X}\right] \ddot{X} = 0$$

No generality is lost in assuming the equations have this form, while considerable simplification is achieved in the derivations that follow.

By differentiating the constraints, and defining the orthogonal projection P onto the range of the transposed constraint matrix

$$P = \left[\frac{\partial \Phi}{\partial X}\right]^T \left(\left[\frac{\partial \Phi}{\partial X}\right]\left[\frac{\partial \Phi}{\partial X}\right]\right)^{-1} \left[\frac{\partial \Phi}{\partial X}\right]$$

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PAGE

the governing equations can be expressed in constraint-reactionless form

$$\ddot{X} + (I-P)KX = (I-P)F$$
$$\ddot{X} + QKX = QF$$

In the above equations,

$$Q = I - P$$

is the orthogonal projection onto the space of admissable configurations.

**(2.2) Undamped Penalty Equations**

While several forms of the penalty equations have appeared in the literature [Arnold], [Bayo 1,2,3], [Kurdila 1,2] and [Park 1], the form chosen here can be derived as in [Bayo] or [Kurdila] from the penalized kinetic and potential energies

$$T = \frac{1}{2}\dot{X}^T\dot{X} + \frac{1}{2}\frac{\dot{\Phi}^T\beta\dot{\Phi}}{\varepsilon}$$

$$T = \frac{1}{2}X^T KX + \frac{1}{2}\frac{\Phi^T\beta\Phi}{\varepsilon}$$

The penalty form of Lagrange's equations

$$L_\varepsilon = T_\varepsilon - V_\varepsilon$$

$$\frac{d}{dt}\left(\frac{\partial L_\varepsilon}{\partial \dot{X}}\right) - \frac{\partial L_\varepsilon}{\partial X} = F$$

then result in

$$\ddot{X}_\varepsilon + KX_\varepsilon = -\frac{1}{\varepsilon}\left[\frac{\partial \Phi}{\partial X}\right]^T \{\dot{\Phi} + \alpha\Phi\} + F$$

$$\ddot{X}_\varepsilon + KX_\varepsilon = -\frac{1}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\ddot{X}_\varepsilon - \frac{\alpha}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]X_\varepsilon + F$$

$$\{I + \frac{1}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}\ddot{X}_\varepsilon + \{K + \frac{\alpha}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}X_\varepsilon = F$$

It is easy to verify *that* [Albert]

$$I + \frac{1}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]$$

is invertible for every e>0, and that

$$\{I + \frac{1}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}^{-1} = \varepsilon\{\varepsilon + \left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}^{-1}$$

This identity allows one to rewrite the governing equations

$$X + \{I + \frac{1}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}^{-1}\{K + \frac{\alpha}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}X_\varepsilon = \{I + \frac{1}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}^{-1}F$$

in the equivalent form

$$\ddot{X} + \varepsilon\{\varepsilon + \left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}^{-1}\{K + \frac{\alpha}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}X_\varepsilon = \varepsilon\{\varepsilon + \left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}^{-1}F$$

By carrying out the multiplications

$$\ddot{X} + \varepsilon \left\{ \varepsilon + \left[\frac{\partial \Phi}{\partial X}\right]^T \left[\frac{\partial \Phi}{\partial X}\right] \right\}^{-1} K X_\varepsilon + \alpha \left\{ \varepsilon + \left[\frac{\partial \Phi}{\partial X}\right]^T \left[\frac{\partial \Phi}{\partial X}\right] \right\}^{-1} \left[\frac{\partial \Phi}{\partial X}\right]^T \left[\frac{\partial \Phi}{\partial X}\right] X =$$

$$\varepsilon \left\{ \varepsilon + \left[\frac{\partial \Phi}{\partial X}\right]^T \left[\frac{\partial \Phi}{\partial X}\right] \right\}^{-1} F$$

the equations can be written in symbolic form as

$$\ddot{X}_\varepsilon + Q(\varepsilon) K X_\varepsilon + \alpha P(\varepsilon) = Q(\varepsilon) F$$

where we have introduced the definitions

$$Q(\varepsilon) = \varepsilon \left\{ \varepsilon + \left[\frac{\partial \Phi}{\partial X}\right]^T \left[\frac{\partial \Phi}{\partial X}\right] \right\}$$

$$P(\varepsilon) = \left\{ \varepsilon + \left[\frac{\partial \Phi}{\partial X}\right]^T \left[\frac{\partial \Phi}{\partial X}\right] \right\}^{-1} \left[\frac{\partial \Phi}{\partial X}\right]^T \left[\frac{\partial \Phi}{\partial X}\right]$$

One should note that

$$I - P(\varepsilon) = Q(\varepsilon)$$

and that

$$P \rightarrow P(\varepsilon)$$
$$Q \rightarrow Q(\varepsilon)$$

in the bounded linear operator topology.

(2.3) **Boundedness of Penalized Solutions For Undamped Case**

Before establishing the convergence of the penalty approximations, it is first necessary to show that the approximations remain bounded

$$\|X_\varepsilon(T)\| \le C_1$$

$$\|\dot{X}_\varepsilon(T)\| \le C_2$$

for any arbitrary, fixed final time T. To this end one can take the inner product of the penalized governing equations with the derivative of the penalized solution to obtain

$$\langle \{I + \frac{1}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}\ddot{X}_\varepsilon, \dot{X}_\varepsilon\rangle + \langle \{K + \frac{1}{\varepsilon}\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\}\ddot{X}_\varepsilon, \dot{X}_\varepsilon\rangle = \langle F, \dot{X}_\varepsilon\rangle$$

This expression can be re-written in a time rate of change of energy form:

$$\frac{d}{dt}\{\frac{1}{2}\|\dot{X}_\varepsilon\| + \langle KX_\varepsilon, X_\varepsilon\rangle\} +$$

$$\frac{d}{dt}\{\frac{1}{2\varepsilon}\dot{X}_\varepsilon^T(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right])\dot{X}_\varepsilon^T + \frac{\alpha}{2\varepsilon}X_\varepsilon^T(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right])X_\varepsilon^T\} = \langle F, X_\varepsilon\rangle$$

A single integration in time yields

$$\frac{1}{2}\|\dot{X}_\varepsilon(t)\|^2 + \langle KX_\varepsilon(t), X_\varepsilon(t)\rangle$$
$$+\left(\frac{1}{2\varepsilon}\dot{X}_\varepsilon^T(t)\left(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\right)\dot{X}_\varepsilon(t) + \frac{\alpha}{2\varepsilon}X_\varepsilon^T(t)\left(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\right)X_\varepsilon(t)\right)$$
$$= \frac{1}{2}\|\dot{X}_\varepsilon(0)\|^2 + \langle KX_\varepsilon(0), X_\varepsilon(0)\rangle$$
$$+\left(\frac{1}{2\varepsilon}\dot{X}_\varepsilon^T(0)\left(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\right)\dot{X}_\varepsilon(0) + \frac{\alpha}{2\varepsilon}X_\varepsilon^T(0)\left(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\right)X_\varepsilon(0)\right)$$
$$+\int_0^t\langle F(\tau), \dot{X}_\varepsilon(\tau)\rangle d\tau$$

But since all penalized equations are required to satisfy the constraints,

$$\left[\frac{\partial\Phi}{\partial X}\right]X_\varepsilon(0) = \left[\frac{\partial\Phi}{\partial X}\right]\dot{X}_\varepsilon(0) = 0$$

the energy expression becomes

$$\frac{1}{2}\|\dot{X}_\varepsilon(t)\|^2 + \langle KX_\varepsilon(t), X_\varepsilon(t)\rangle$$
$$+\left(\frac{1}{2\varepsilon}\dot{X}_\varepsilon^T(t)\left(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\right)\dot{X}_\varepsilon(t) + \frac{\alpha}{2\varepsilon}X_\varepsilon^T(t)\left(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right]\right)X_\varepsilon(t)\right)$$
$$= \frac{1}{2}\|\dot{X}_\varepsilon(0)\|^2 + \langle KX_\varepsilon(0), X_\varepsilon(0)\rangle$$
$$+\int_0^t\langle F(\tau), \dot{X}_\varepsilon(\tau)\rangle d\tau$$

In particular, this implies that we can write the inequality

$$\frac{1}{2}\|\dot{X}_\varepsilon(t)\|^2 + \langle KX_\varepsilon(t), X_\varepsilon(t)\rangle$$
$$\leq \frac{1}{2}\|\dot{X}_\varepsilon(0)\|^2 + \langle KX_\varepsilon(0), X_\varepsilon(0)\rangle$$
$$+ \int_0^t \|F(\tau)\|\|\dot{X}_\varepsilon(\tau)\|d\tau$$

But by using the (trivial) form of Minkowski's inequality [Oden]

$$2|ab| \leq a^2 + b^2$$

one can write

$$\frac{1}{2}\|\dot{X}_\varepsilon(t)\|^2 + \langle KX_\varepsilon(t), X_\varepsilon(t)\rangle$$
$$\leq \frac{1}{2}\|\dot{X}_\varepsilon(0)\|^2 + \frac{1}{2}\langle KX_\varepsilon(0), X_\varepsilon(0)\rangle$$
$$+ \frac{1}{2}\int_0^t \|F(\tau)\|^2 d\tau + \frac{1}{2}\int_0^t \|\dot{X}_\varepsilon(\tau)\|^2 d\tau$$

$$\frac{1}{2}\|\dot{X}_\varepsilon(t)\|^2 + \frac{1}{2}\langle KX_\varepsilon(t), X_\varepsilon(t)\rangle$$
$$\leq \{\frac{1}{2}\|\dot{X}_\varepsilon(0)\|^2 + \frac{1}{2}\langle KX_\varepsilon(0), X_\varepsilon(0)\rangle + \frac{1}{2}\int_0^t \|F(\tau)\|^2 d\tau\}$$
$$+ \frac{1}{2}\int_0^t \|\dot{X}_\varepsilon(\tau)\|^2 d\tau$$

Finally, one can write the energy expression in a form appropriate for the application of Gronwall's inequality [Wloka]

$$\frac{1}{2}\|\dot{X}_\varepsilon(t)\|^2 + \frac{1}{2}\langle KX_\varepsilon(t), X_\varepsilon(t)\rangle$$
$$\leq \{\frac{1}{2}\|\dot{X}_\varepsilon(0)\|^2 + \frac{1}{2}\langle KX_\varepsilon(0), X_\varepsilon(0)\rangle + \frac{1}{2}\int_0^t \|F(\tau)\|^2 d\tau\}$$
$$+ \frac{1}{2}\int_0^t (\|\dot{X}_\varepsilon(\tau)\|^2 + \langle KX_\varepsilon(\tau), X_\varepsilon(\tau)\rangle)\, d\tau$$

$$\frac{1}{2}\|\dot{X}_\varepsilon(t)\|^2 + \frac{1}{2}\langle KX_\varepsilon(t), X_\varepsilon(t)\rangle \leq g(t) + \frac{1}{2}\int_0^t (\|\dot{X}_\varepsilon(\tau)\|^2 + \frac{1}{2}\langle KX_\varepsilon(\tau), X_\varepsilon(\tau)\rangle)\, d\tau$$

Application of Gronwall's inequality to the above expression implies the desired result that the penalized solution remains bounded

$$\|\dot{X}_\varepsilon(t)\| \leq \kappa_1$$

$$\langle KX_\varepsilon(t), X_\varepsilon(t)\rangle \leq \kappa_2$$

for a fixed time t, 0<t<T.

**(2.4) Variational Dependence of Error on Constraint Violation**

With the above result implying that the solution of the penalty equations remains bounded, one can now derive a relationship between the error in the penalty approximation

$$\|X - X_\varepsilon\|$$

and the norm of the constraint violation

$$\|PX_{\varepsilon}\|^2 = \left\| \Phi_{\varepsilon}^T \left( \left[\frac{\partial\Phi}{\partial X}\right] \left[\frac{\partial\Phi}{\partial X}\right] \right)^{-1} \Phi_{\varepsilon} \right\|$$

Subtracting the original, exact equations and the penalty form of the equations, one can obtain

$$\begin{aligned} &\ddot{X} - \ddot{X}_{\varepsilon} + QKX - Q(\varepsilon)KX_{\varepsilon} \\ &= (Q - Q(\varepsilon))F + \alpha P(\varepsilon)X_{\varepsilon} \end{aligned}$$

By the addition and subtraction of identical terms, and by employing the triangle inequality, one can write that

$$\begin{aligned} &\|\ddot{X} - \ddot{X}_{\varepsilon}\| + \|Q - Q(\varepsilon)\|\|KX_{\varepsilon}\| + \|Q\|\|K(X - X_{\varepsilon})\| \\ &\le \|Q - Q(\varepsilon)\|\|F\| + \alpha\|PX_{\varepsilon}\| + \alpha\|P(\varepsilon) - P\|\|X_{\varepsilon}\| \end{aligned}$$

Using the uniform convergence of $P(\varepsilon)$ to P, the fact that the solution of the penalty equations remains bounded, the final variational error inequality is achieved.

$$\lim_{\varepsilon \to 0} \{\|\ddot{X} - \ddot{X}_{\varepsilon}\| + \|K(X - X_{\varepsilon})\|\} \le \alpha\|PX_{\varepsilon}\| \le \alpha_1\|\Phi_{\varepsilon}\|$$

## (3.0) LINEAR, DAMPED MDOF EQUATIONS

### (3.1) Original, Exact Equations

The derivation of a corresponding variational statement for the case of damped, MDOF systems follows in much the same manner as the strategy empolyed in the undamped case. In this section, the governing system of differential-algebraic equations are

$$\hat{M}\ddot{x} + \hat{C}\dot{x} + \hat{K}x = \left[\frac{\partial\Phi}{\partial x}\right]^T \lambda + \hat{F}$$

$$\left[\frac{\partial\Phi}{\partial x}\right] x = 0$$

where, as in the undamped case,

$$x \in R^N \qquad \hat{F} \in R^N$$
$$\Phi \in R^D \qquad \hat{M} \in R^{N \times N}$$
$$\lambda \in R^D \qquad \hat{K} \in R^{N \times N}$$
$$\left[\frac{\partial\Phi}{\partial x}\right] \in R^{D \times N} \qquad \hat{C} \in R^{N \times N}$$

The same assumptions regarding the properties of the mass, stiffness and constraint matrices are employed in the following arguments, as well as the stipulation that the damping matrix is positive semi-definite

$$\langle \hat{C}x, x \rangle \geq 0$$

With the identical change of coordinates employed in the undamped equations, one can consider the system defined below without loss of generality:

$$C = \hat{M}^{-1/2}\hat{C}\hat{M}^{-1/2}$$

$$\ddot{X} + C\dot{X} + KX = \left[\frac{\partial\Phi}{\partial X}\right]^T \lambda + F$$

$$\left[\frac{\partial\Phi}{\partial X}\right] X = 0$$

The new system of reaction-free equations now contain a term representing viscous damping

$$\ddot{X} + (I-P)C\dot{X} + (I-P)KX = (I-P)F$$

$$\ddot{X} + QC\dot{X} + QKX = QF$$

## (3.2) Damped Penalty Equations

The penalty formulation employed in the case of a damped, linear MDOF system has been selected as in [Bayo] and [Kurdila]. In addition to the penalized kinetic and potential energies, a generalized Rayleight dissipation function is introduced

$$T_\varepsilon = \frac{1}{2}\dot{X}^T\dot{X} + \frac{1}{2}\frac{\dot{\Phi}^T\beta\dot{\Phi}}{\varepsilon}$$

$$V_\varepsilon = \frac{1}{2}X^T KX + \frac{1}{2}\frac{\Phi^T\beta\Phi}{\varepsilon}$$

$$F_\varepsilon = \frac{1}{2}\dot{X}^T C\dot{X} + \frac{1}{2}\frac{\dot{\Phi}^T\mu\dot{\Phi}}{\varepsilon}$$

The penalized form of Lagrange's equations is consequently

$$L_{\varepsilon} = T_{\varepsilon} - V_{\varepsilon}$$

$$\frac{d}{dt}\left(\frac{\partial L_{\varepsilon}}{\partial \dot{X}}\right) - \frac{\partial L_{\varepsilon}}{\partial X} + \frac{\partial F_{\varepsilon}}{\partial \dot{X}} = F$$

The viscously damped version of the feedback form of the penalty equations are

$$\ddot{X}_{\varepsilon} + C\dot{X}_{\varepsilon} + KX_{\varepsilon} = -\frac{1}{\varepsilon}\left[\frac{\partial \Phi}{\partial X}\right]^{T} \{\ddot{\Phi} + \mu\dot{\Phi} + \alpha\Phi\} + F$$

$$\{I + \frac{1}{\varepsilon}\left[\frac{\partial \Phi}{\partial X}\right]^{T}\left[\frac{\partial \Phi}{\partial X}\right]\}\ddot{X}_{\varepsilon} + \{C + \frac{\alpha}{\varepsilon}\left[\frac{\partial \Phi}{\partial X}\right]^{T}\left[\frac{\partial \Phi}{\partial X}\right]\}\dot{X}_{\varepsilon} + \{K + \frac{\alpha}{\varepsilon}\left[\frac{\partial \Phi}{\partial X}\right]^{T}\left[\frac{\partial \Phi}{\partial X}\right]\}X_{\varepsilon} = F$$

$$\ddot{X}_{\varepsilon} + Q(\varepsilon)C\dot{X}_{\varepsilon} + Q(\varepsilon)KX_{\varepsilon} + \mu P(\varepsilon)\dot{X}_{\varepsilon} + \alpha P(\varepsilon)X_{\varepsilon} = Q(\varepsilon)F$$

**(3.3) Boundedness of the Solutions of the Damped Penalty Equations**

Again before estimating the error in the approximation, as in section (2), it is necessary to establish that the solution of the penalty equations corresponding to damped system is bounded

$$\|X_{\varepsilon}(t)\| \le C_1$$

$$\|\dot{X}_{\varepsilon}(t)\| \le C_2$$

for any arbitrary, finite time 0<t<T.

Taking the inner product of the damped penalty equations and the derivative of the penalized solution, one can write a time-rate-of-change of energy as carried out in the undamped analysis.

$$\frac{d}{dt}\{\frac{1}{2}\|\dot{X}_\varepsilon\| + \langle KX_\varepsilon, X_\varepsilon\rangle\} +$$

$$\frac{d}{dt}\{\frac{1}{2\varepsilon}\dot{X}_\varepsilon^T(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right])\dot{X}_\varepsilon + \frac{\alpha}{2\varepsilon}X_\varepsilon^T(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right])X_\varepsilon\} +$$

$$\langle CX_\varepsilon, X_\varepsilon\rangle + \frac{\mu}{2\varepsilon}\dot{X}_\varepsilon^T(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right])\dot{X}_\varepsilon = \langle F, \dot{X}_\varepsilon\rangle$$

Because the damping matrix C is symmetric, positive semi-definite, one can write the inequality

$$\frac{d}{dt}\{\frac{1}{2}\|\dot{X}_\varepsilon\| + \langle KX_\varepsilon, X_\varepsilon\rangle\} +$$

$$\frac{d}{dt}\{\frac{1}{2\varepsilon}\dot{X}_\varepsilon^T(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right])\dot{X}_\varepsilon + \frac{\alpha}{2\varepsilon}X_\varepsilon^T(\left[\frac{\partial\Phi}{\partial X}\right]^T\left[\frac{\partial\Phi}{\partial X}\right])X_\varepsilon\} \le \langle F, \dot{X}_\varepsilon\rangle$$

Following precisely the same steps as in the analysis of the undamped case, one integration yields

$$\frac{1}{2}\|\dot{X}_\varepsilon(t)\|^2 + \frac{1}{2}\langle KX_\varepsilon(t), X_\varepsilon(t)\rangle$$

$$\le \{\frac{1}{2}\|\dot{X}_\varepsilon(0)\|^2 + \frac{1}{2}\langle KX_\varepsilon(0), X_\varepsilon(0)\rangle + \frac{1}{2}\int_0^t\|F(\tau)\|^2 d\tau\}$$

$$+\frac{1}{2}\int_0^t(\|\dot{X}_\varepsilon(\tau)\|^2 + \langle KX_\varepsilon(\tau), X_\varepsilon(\tau)\rangle)\, d\tau$$

$$\frac{1}{2}\|\dot{X}_\varepsilon(t)\|^2 + \frac{1}{2}\langle KX_\varepsilon(t), X_\varepsilon(t)\rangle \le g(t) + \frac{1}{2}\int_0^t(\|\dot{X}_\varepsilon(\tau)\|^2 + \frac{1}{2}\langle KX_\varepsilon(\tau), X_\varepsilon(\tau)\rangle)\, d\tau$$

which enables one to apply Gronwall's inequality to conclude

$$\|\dot{X}_\varepsilon(t)\| \leq \kappa_1$$

$$\langle KX_\varepsilon(t), X_\varepsilon(t)\rangle \leq \kappa_2$$

for a fixed time 0<t<T.

**(3.4) Variational Dependence of Error on Constraint Violation**

The final variational relationship between the constraint violation and the approximation error can now be achieved by subtracting the exact, reactionless equations and the penalized equations.

$$\begin{aligned}\ddot{X} - \ddot{X}_\varepsilon + QC\dot{X} - Q(\varepsilon)C\dot{X}_\varepsilon \\ + QKX - Q(\varepsilon)KX_\varepsilon \\ = (Q - Q(\varepsilon))F + \mu P(\varepsilon)X_\varepsilon + \alpha P(\varepsilon)X_\varepsilon\end{aligned}$$

As in the analysis of the undamped equations, the addition and subtraction of identical terms and the application of the triangle inequality enables one to write

$$\begin{aligned}\|\ddot{X} - \ddot{X}_\varepsilon\| + \|Q - Q(\varepsilon)\|\|C\dot{X}_\varepsilon\| + \|Q\|\|C(\dot{X} - \dot{X}_\varepsilon)\| \\ + \|Q - Q(\varepsilon)\|\|KX_\varepsilon\| + \|Q\|\|K(X - X_\varepsilon)\| \\ \leq \|Q - Q(\varepsilon)\|\|F\| + \alpha\|PX_\varepsilon\| + \alpha\|P(\varepsilon) - P\|\|X_\varepsilon\| \\ + \mu\|PX_\varepsilon\| + \mu\|P(\varepsilon) - P\|\|\dot{X}_\varepsilon\|\end{aligned}$$

In the limit as ε approaches 0, one can use the fact that

$$Q(\varepsilon) \to Q$$
$$P(\varepsilon) \to P$$

in operator norm and the boundedness of Xe to obtain

$$\lim_{\varepsilon \to 0} \{ \|\dot{X} - \dot{X}_\varepsilon\| + \|C(\dot{X} - \dot{X}_\varepsilon)\| + \|K(X - X_\varepsilon)\| \} \le \alpha \|PX_\varepsilon\| + \mu \|PX_\varepsilon\| \le \alpha_1 \|\Phi_\varepsilon\| + \mu_1 \|\dot{\Phi}_\varepsilon\|$$

## (4.0) CONSERVATIVE FORCING FUNCTIONS

The variational error bounds discussed so far have special significance when the forcing funtion F is in fact conservative. When this is the case, one can write [Kurdila]

$$E_\varepsilon(t) = E(0) - \int_0^t \{ \langle C\dot{X}_\varepsilon, \dot{X}_\varepsilon \rangle + \frac{\mu}{2\varepsilon} \dot{X}_\varepsilon^T \left( \left[\frac{\partial \Phi}{\partial X}\right]^T \left[\frac{\partial \Phi}{\partial X}\right] \right) \dot{X}_\varepsilon \} \, d\tau$$

where

$$E_\varepsilon(t) = \frac{1}{2} \dot{X}_\varepsilon^T M \dot{X}_\varepsilon + \frac{1}{2} X_\varepsilon^T K X_\varepsilon$$

$$+ \frac{1}{2\varepsilon} \dot{\Phi}_\varepsilon^T \dot{\Phi}_\varepsilon + \frac{1}{2\varepsilon} \Phi_\varepsilon^T \alpha \Phi_\varepsilon$$

$$E(0) = \frac{1}{2} \dot{X}_0^T M \dot{X}_0 + \frac{1}{2} X_0^T K X_0$$

As shown in [Kurdial 1,2,3], one has the bounds

$$\|\dot{\Phi}_\varepsilon\| + \|\Phi_\varepsilon\| \le \kappa_1 \varepsilon E(0)$$

for some constant $\kappa$. This inequality implies that

$$\|C(\dot{X} - \dot{X}_\varepsilon)\| + \|K(X - X_\varepsilon)\| \le \kappa_2 (\varepsilon E(0))^{\frac{1}{2}}$$

which provides a rate of convergence when the forcing term is conservative.

## (5.0) CONCLUSIONS

By using standard analyses from the field of partial differential equations and linear regression, innovative variational estimates have been derived that allow one to monitor the fidelity of penalty method approximations. The analysis herein is applied to linear systems and guarantees convergence rates for conservative forcing functions. The results provide a qualitative distinction between the linear and nonlinear cases: convergence of constraint norm to zero implies convergence of the method in the linear case, while bifurcation phenomenon preclude a similar conclusion in nonlinear simulations, unless additional assumptions regarding regularity are made.

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